Proof of the Borde-Guth-Vilenkin theorem

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$x^\mu = (ct, x^i)$ $\qquad \mu = 0, 1, 2, 3$

$g_{ij} = a^2 \gamma_{ij}$ $\qquad i, j = 1, 2, 3$

Based on:
Borde, A., Guth, A. H., Vilenkin, A.
PRL 90, 15, 151301 (2003)

Geodesic equation:

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = \emptyset$$

$\mu = \emptyset$:

$$c \frac{d^2 t}{d\lambda^2} + \Gamma^\emptyset_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = \emptyset$$

$$\rightarrow \Gamma^\emptyset_{\emptyset\emptyset} = \emptyset = \Gamma^\emptyset_{\emptyset i} = \Gamma^\emptyset_{i\emptyset}$$

$$\rightarrow \Gamma^\emptyset_{ij} = -\frac{1}{2} g^{\emptyset\emptyset} \frac{\partial g_{ij}}{\partial(ct)} = \left(-\frac{1}{2}\right)(-1) \frac{\partial a^2}{\partial(ct)} \gamma_{ij} = a\dot{a} \frac{1}{c} \gamma_{ij}$$

$$\Rightarrow \underbrace{c^2 \frac{d^2 t}{d\lambda^2} + a\dot{a} \gamma_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = \emptyset} \qquad (1)$$

$$\rightarrow ds^2 = -c^2 dt^2 + a^2 \gamma_{ij} dx^i dx^j$$

For null geodesics:

$$\underbrace{c^2 \frac{dt^2}{d\lambda^2} - a^2 \gamma_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = \emptyset} \qquad (2)$$

$a \cdot (1) + \dot{a}(2)$:

$$a \cancel{c^2} \frac{d^2 t}{d\lambda^2} + \dot{a} \cancel{c^2} \frac{dt^2}{d\lambda^2} = \emptyset$$

$$a(\lambda) \frac{d^2 t}{d\lambda^2} + \frac{da}{d\lambda} \cdot \frac{\cancel{d\lambda}}{\cancel{dt}} \cdot \frac{dt^{\cancel{2}}}{d\lambda^{\cancel{2}}} = \emptyset \qquad \frac{dt}{d\lambda} \overset{!}{=} u$$

$$a \frac{du}{d\lambda} + \frac{da}{d\lambda} \cdot u = \emptyset$$

$$\frac{d}{d\lambda}(au) = \emptyset \Rightarrow au = \text{const.} = a_0$$

$$\frac{dt}{d\lambda} = \frac{a_0}{a} \Rightarrow \boxed{d\lambda = \frac{a}{a_0} dt}$$

For light: $k^\mu = \frac{dx^\mu}{d\lambda} \cdot \text{const.}$

$$k^\emptyset = \underline{\underline{\omega}} = \frac{dt}{d\lambda} \cdot \text{const} \sim \underline{\underline{\frac{1}{a}}}$$

(2.)

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$\rightarrow ds^2 = -c^2 dt^2 + a^2 \gamma_{ij} dx^i dx^j$

For timelike geodesics: $ds^2 = -c^2 d\tau^2$

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \stackrel{!}{=} u$$

$$\underbrace{c^2 \frac{dt^2}{d\tau^2} - c^2 - a^2 \gamma_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = \emptyset}_{} \quad (2)$$

$a \cdot (1) + \dot{a}(2)$:

$$a \frac{d^2 t}{d\tau^2} + \dot{a}\left(\frac{dt^2}{d\tau^2} - 1\right) = \emptyset$$

$$a \frac{du}{d\tau} + \frac{da}{d\tau}\left(u - \frac{1}{u}\right) = \emptyset$$

$$\frac{u}{u^2-1} du = -\frac{da}{a}$$

$$d\left(\log\sqrt{u^2-1}\right) = d\left(\log\frac{1}{a}\right)$$

$$\sqrt{u^2-1} = \frac{a_0}{a}$$

$$\frac{1}{1-\frac{v^2}{c^2}} - 1 = \left(\frac{a_0}{a}\right)^2$$

$$\underbrace{\frac{\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} = \left(\frac{a_0}{a}\right)^2}_{} \Rightarrow a = 1 \rightarrow v = v_0$$

$$\Downarrow$$

$$a_0^2 = \frac{\frac{v_0^2}{c^2}}{1-\frac{v_0^2}{c^2}}$$

$$\Rightarrow \boxed{\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{v_0}{\sqrt{1-\frac{v_0^2}{c^2}}} \frac{1}{a}}$$

For $v, v_0 \ll c$: $v \simeq \frac{v_0}{a}$

For $a \rightarrow \emptyset \Rightarrow v \rightarrow c$

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(1) Null geodesics:

$$\lambda_f = \lambda(t_f) \qquad a(t_f) = a_0$$

$$\lambda_i = \lambda(t_i)$$

$$\int_{\lambda_i}^{\lambda_f} H(\lambda)d\lambda = \int_{\lambda_i}^{\lambda_f} \frac{\dot{a}}{a} d\lambda = \int_{\lambda_i}^{\lambda_f} \frac{1}{a}\frac{da}{d\lambda}\frac{d\lambda}{dt} d\lambda = \int_{\lambda_i}^{\lambda_f} \frac{1}{\cancel{a}}\frac{da}{d\lambda}\frac{\cancel{a}}{a_0} d\lambda = \int_{a_i}^{a_0} \frac{da}{a_0} = 1 - \frac{a_i}{a_0}$$

$$\int_{\lambda_i}^{\lambda_f} H(\lambda)d\lambda \leqslant 1$$

$$\underbrace{\frac{1}{\lambda_f - \lambda_i}\int_{\lambda_i}^{\lambda_f} H(\lambda)d\lambda}_{H_{av}} \leqslant \frac{1}{\lambda_f - \lambda_i}$$

If $+\infty > H_{av} > \emptyset \Rightarrow \frac{1}{H_{av}} \geqslant \lambda_f - \lambda_i \Rightarrow \boxed{+\infty > \lambda_f - \lambda_i}$

(2) Timelike geodesics: $u = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\int_{J_i}^{J_f} H(J)dJ = \int_{J_i}^{J_f} \frac{1}{a}\frac{da}{dJ}\frac{dJ}{dt} dJ = -\int_{\gamma_i}^{\gamma} \frac{\cancel{u}}{u^2-1}\frac{1}{\cancel{u}} du = \left[\frac{1}{2}\ln\frac{u+1}{u-1}\right]_{\gamma_i}^{\gamma} =$$

$$= \frac{1}{2}\ln\frac{\gamma+1}{\gamma-1} - \underbrace{\frac{1}{2}\ln\frac{\gamma_i+1}{\gamma_i-1}}_{+\infty > \dots \geqslant \emptyset} \leqslant \underbrace{\frac{1}{2}\ln\frac{\gamma+1}{\gamma-1}}_{+\infty > \dots \geqslant \emptyset} = \text{const.}$$

$(+\infty > \gamma_i \geqslant 1)$

$$H_{av} = \frac{1}{J_f - J_i}\int_{J_i}^{J_f} H(J)dJ \leqslant \frac{\text{const.}}{J_f - J_i}$$

If $+\infty > H_{av} > \emptyset \Rightarrow \frac{1}{H_{av}} \geqslant \frac{J_f - J_i}{\text{const.}} \Rightarrow \frac{\text{const.}}{H_{av}} \geqslant J_f - J_i \Rightarrow \boxed{+\infty > J_f - J_i}$