E.o.S. for relativistic particles

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Let us consider a gas of bosons and fermions in thermal equilibrium at temperature T.

The phase space density of these particles will be:

$$f(\underline{q}) = \frac{1}{e^{\frac{E(\underline{q})-\mu}{k_B T}} \pm 1}$$

where q is the momentum, and μ is the chemical potential.

$\ominus$ Bose-Einstein distribution for bosons.

$\oplus$ Fermi-Dirac distribution for fermions

Also:

$$E(\underline{q}) = \sqrt{q^2c^2 + m_0^2c^4} \qquad q = |\underline{q}|$$

Thus

Number density: $$n = \frac{g}{(2\pi\hbar)^3}\int d^3\underline{q}\, f(\underline{q})$$

Mass density: $$\rho = \frac{g}{(2\pi\hbar)^3}\frac{1}{c^2}\int d^3\underline{q}\, E(\underline{q}) f(\underline{q})$$

g: degrees of freedom

$g = n_{types} \cdot n_{antiparticles} \cdot n_{spin}$

$g_\gamma = 1 \cdot 1 \cdot 2 = 2$

$g_\nu = 3 \cdot 2 \cdot 1 = 6$

$$\Rightarrow \int d^3\underline{q} \,\sqcup = 4\pi \int_0^{+\infty} dq\, q^2 \,\sqcup$$

$$n = \frac{g}{2\pi^2\hbar^3}\int_0^{+\infty} dq \frac{q^2}{e^{\frac{E(q)-\mu}{k_B T}} \pm 1}$$

$$\rho = \frac{g}{2\pi^2\hbar^3c^2}\int_0^{+\infty} dq \frac{q^2\sqrt{q^2c^2 + m_0^2c^4}}{e^{\frac{E(q)-\mu}{k_B T}} \pm 1}$$

For relativistic particles: $qc \gg m_0c^2 \gg \mu \Rightarrow E(q) \simeq qc$

$$n = \frac{g}{2\pi^2\hbar^3}\int_0^{+\infty} dq \frac{q^2}{e^{\frac{qc}{k_B T}} \pm 1}$$

$$x = \frac{qc}{k_B T} \Rightarrow q = \frac{k_B T}{c} x \qquad dq = \frac{k_B T}{c} dx$$

$$n = \frac{g}{2\pi^2\hbar^3}\left(\frac{k_B T}{c}\right)^3 \int_0^{+\infty} dx \frac{x^2}{e^x \pm 1} = \underline{\underline{\frac{g'}{\pi^2\hbar^3}\left(\frac{k_B T}{c}\right)^3 \zeta(3)}}$$

$$g' = \sum_{i=1}^{N} g_i \alpha_i$$

$\alpha_i = 1$ for bosons

$\alpha_i = \frac{3}{4}$ for fermions

$\zeta(3) \approx 1.202$ Riemann zeta function

$$\rho = \frac{g}{2\pi^2\hbar^3 c}\int_0^{+\infty} dq \frac{q^3}{e^{\frac{qc}{k_B T}} \pm 1} = \frac{g}{2\pi^2\hbar^3 c}\left(\frac{k_B T}{c}\right)^4 \int_0^{+\infty} dx \frac{x^3}{e^x \pm 1}$$

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$$\rho = \frac{g_* \pi^2}{30 \hbar^3 c} \left(\frac{k_B T}{c}\right)^4 \qquad g_* = \sum_{i=1}^{N} g_i \beta_i$$

$\beta_i = 1$ for bosons

$\beta_i = \frac{7}{8}$ for fermions

Deriving pressure

(a) Consider a particle bouncing back and forth between two walls parallel to the yz-plane separated by length L_x. A momentum change of $2q_x$ occurs in every collision.

$$\left.\begin{aligned} q_x &= \gamma m_0 v_x \\ E &= \gamma m_0 c^2 \end{aligned} \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \right\} v_x = \frac{c^2 q_x}{E}$$

The time between collisions is $\Delta t = \frac{2L_x}{v_x} = \frac{2L_x E}{c^2 q_x}$

Then the average force on the wall is:

$$\bar{F} = \frac{\Delta q_x}{\Delta t} = \frac{2q_x}{2L_x E} \cdot c^2 q_x = \frac{1}{L_x} \frac{c^2 q_x^2}{E}$$

Dividing both sides by the $A = L_y L_z$ area of the wall, we get the pressure:

$$p = \frac{1}{V} \frac{c^2 q_x^2}{E}$$

Using the relation for averages: $\langle q^2 \rangle = \langle q_x^2 \rangle + \langle q_y^2 \rangle + \langle q_z^2 \rangle = 3 \langle q_x^2 \rangle$

$$\Rightarrow q_x^2 = \frac{q^2}{3}$$

Summing over all particles in phase space:

$$p = \frac{g}{(2\pi\hbar)^3} \int d^3q \, \frac{c^2 q^2}{3E} f(q) = \frac{g}{2\pi^2 \hbar^3} \int_0^{+\infty} dq \, \frac{c^2 q^4}{3\sqrt{q^2 c^2 + m_0^2 c^4}} \frac{1}{e^{\frac{E(q)-\mu}{k_B T}} \pm 1}$$

For relativistic particles: $qc \gg m_0 c^2 \gg \mu$

$$p = \frac{g c}{6\pi^2 \hbar^3} \int_0^{+\infty} dq \, \frac{q^3}{e^{\frac{qc}{k_B T}} \pm 1} = \frac{g c}{6\pi^2 \hbar^3} \left(\frac{k_B T}{c}\right)^4 \int_0^{+\infty} dx \, \frac{x^3}{e^x \pm 1}$$

$$p = \frac{1}{3} c^2 \frac{g_* \pi^2}{30 \hbar^3 c} \left(\frac{k_B T}{c}\right)^4 \qquad g_* = \sum_{i=1}^{N} g_i \beta_i$$

$\beta_i = 1$ for bosons

$\beta_i = \frac{7}{8}$ for fermions

$$p = \frac{1}{3} \rho c^2 \Rightarrow \boxed{w = \frac{1}{3}}$$

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(b) The first law of thermodynamics

$$dU = TdS - pdV + \sum_i \mu_i dN_i$$

$U = \rho c^2 V \qquad S = sV \qquad \mu_i \simeq 0 \qquad s = s(T) \qquad \rho = \rho(T)$

$$d(\rho c^2 V) = Td(sV) - pdV$$

$$\rho c^2 dV + c^2 V d\rho = TsdV + TVds - pdV$$

$$\rho c^2 dV + c^2 V \frac{d\rho}{dT} dT = TsdV + TV\frac{ds}{dT}dT - pdV$$

↳ (1) $\rho c^2 = Ts - p \quad \Rightarrow \quad s = \dfrac{\rho c^2 + p}{T}$

$$p = Ts - \rho c^2$$

↳ (2) $c^2 \cancel{V} \dfrac{d\rho}{dT} = T\cancel{V}\dfrac{ds}{dT} \qquad \rho = \eta T^4 \qquad \eta = \dfrac{g_* \pi^2}{30\hbar^3 c}\left(\dfrac{k_B}{c}\right)^4$

$$c^2 \cdot 4\eta T^3 = T\frac{ds}{dT}$$

$$\frac{ds}{dT} = 4\eta c^2 T^2 \quad \Rightarrow \quad s(T) = \frac{4}{3}\eta T^3 c^2 \quad (+\,\text{const.} = 0)$$

$$\Rightarrow p = Ts - \rho c^2 = \frac{4}{3}\eta T^4 c^2 - \rho c^2 = \frac{1}{3}\rho c^2 \quad \Rightarrow \quad \boxed{w = \frac{1}{3}}$$