# FLRW geometries

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$$x = (x^0, x^1, x^2, x^3) = (ct, r, \theta, \varphi)$$

FLRW metric:

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & \frac{a^2}{1-kr^2} & & \\ & & a^2r^2 & \\ & & & a^2r^2\sin^2\theta \end{pmatrix} \Rightarrow ds^2 = -c^2dt^2 + a^2d\sigma^2$$

$$[k] = \frac{1}{m^2}$$

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2$$

If $k \neq \emptyset \Rightarrow \boxed{\tilde{k} = \frac{k}{|k|} = \pm 1}$

$$\rightarrow \frac{dr^2}{1-kr^2} = \frac{dr^2}{1-\tilde{k}|k|r^2} = \frac{dr^2}{1-\tilde{k}(\sqrt{|k|}r)^2} \Rightarrow \tilde{r} = \sqrt{|k|}r \Rightarrow \frac{1}{\sqrt{|k|}}d\tilde{r} = dr$$

$$a^2d\sigma^2 = \frac{a^2}{|k|}\left(\frac{d\tilde{r}^2}{1-\tilde{k}\tilde{r}^2} + \tilde{r}^2d\theta^2 + \tilde{r}^2\sin^2\theta d\varphi^2\right)$$

$\hookrightarrow R^2(t) = \frac{a^2(t)}{|k|}$ $\quad([R^2(t)] = m^2)$ $\quad R(t)$: radius of curvature

$$R(t) = a(t)\frac{1}{\sqrt{|k|}} = a(t)R_0$$

$\hookrightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varrho - \frac{kc^2}{a^2}$ $\qquad \frac{kc^2}{a^2H^2} = \Omega - 1$

$H^2 = \frac{8\pi G}{3}\varrho - \frac{kc^2}{a^2}$ $\qquad k = \frac{a^2H^2}{c^2}(\Omega - 1)$

$1 = \underbrace{\frac{8\pi G}{3H^2}\varrho}_{\Omega = \frac{\varrho}{\varrho_c}} - \frac{kc^2}{a^2H^2}$ $\qquad k = \frac{a_0^2H_0^2}{c^2}(\Omega_0 - 1)$

$$R(t) = \frac{a(t)}{\sqrt{|k|}} = \frac{c}{H}\frac{1}{\sqrt{|\Omega - 1|}}$$

$$R_0 = \frac{c}{H_0}\frac{1}{\sqrt{|\Omega_0 - 1|}}$$

Only $R_0 = \frac{a_0}{\sqrt{|k|}}$ can be measured
- $\rightarrow$ Physical convention: $a_0 \stackrel{!}{=} 1$
- $\rightarrow$ Geometrical convention: $|k| \stackrel{!}{=} 1\,\frac{1}{m^2}$

$$\Rightarrow d\sigma^2 = R_0^2\left(\frac{d\tilde{r}^2}{1-\tilde{k}\tilde{r}^2} + \tilde{r}^2d\theta^2 + \tilde{r}^2\sin^2\theta d\varphi^2\right)$$

1) $\tilde{k} \stackrel{!}{=} +1 \rightarrow$ Howard P. Robertson (US) + Arthur G. Walker (GB), 1935-37:
„The spatial part of the FLRW metric describes the 3D surface of a 4-ball (3-sphere)."

# FLRW geometries

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Cartesian coordinates in 4D space: $x, y, z, u$

Constraint relation: $x^2+y^2+z^2+u^2=R_0^2$

Natural coordinates (satisfying the constraint): $\chi, \theta, \varphi$

$$\left.\begin{aligned} x &= R_0 \sin\chi \sin\theta \cos\varphi \\ y &= R_0 \sin\chi \sin\theta \sin\varphi \\ z &= R_0 \sin\chi \cos\theta \\ u &= R_0 \cos\chi \end{aligned}\right\} d\sigma^2 = dx^2+dy^2+dz^2+du^2 \quad (+\text{constraint})$$

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$$dx = R_0\cos\chi\sin\theta\cos\varphi\, d\chi + R_0\sin\chi\cos\theta\cos\varphi\, d\theta - R_0\sin\chi\sin\theta\sin\varphi\, d\varphi \rightarrow dx^2 = \dots$$
$$dy = R_0\cos\chi\sin\theta\sin\varphi\, d\chi + R_0\sin\chi\cos\theta\sin\varphi\, d\theta + R_0\sin\chi\sin\theta\cos\varphi\, d\varphi \rightarrow dy^2 = \dots$$
$$dz = R_0\cos\chi\cos\theta\, d\chi - R_0\sin\chi\sin\theta\, d\theta \rightarrow dz^2 = \dots$$
$$du = -R_0\sin\chi\, d\chi \rightarrow du^2 = \dots$$

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$$d\sigma^2 = dx^2+dy^2+dz^2+du^2 = R_0^2\left(d\chi^2 + \sin^2\chi\, d\theta^2 + \sin^2\chi\sin^2\theta\, d\varphi^2\right)$$
(+ constraint)

$$\tilde{r} \overset{!}{=} \sin\chi$$
$$d\tilde{r} = \cos\chi\, d\chi = \sqrt{1-\sin^2\chi}\, d\chi = \sqrt{1-\tilde{r}^2}\, d\chi \Rightarrow d\chi = \frac{d\tilde{r}}{\sqrt{1-\tilde{r}^2}}$$

$$\Rightarrow d\sigma^2 = R_0^2\left(\frac{d\tilde{r}^2}{1-\tilde{r}^2} + \tilde{r}^2 d\theta^2 + \tilde{r}^2\sin^2\theta\, d\varphi^2\right) \quad \checkmark$$

* $\Rightarrow$ Volume of the universe today: $V = S_3 = 2\pi^2 R_0^3 = 2\pi^2 \frac{c^3}{H_0^3}|\Omega_0 - 1|^{-\frac{3}{2}}$

Volume of the observable universe: $V_{obs} = \frac{4(\alpha d_H)^3\pi}{3}$ where $\alpha \simeq 3.23$, $d_H = \frac{c}{H_0}$

Ratio: $\frac{V}{V_{obs}} = \frac{2\pi^2 \frac{c^3}{H_0^3}|\Omega_0-1|^{-\frac{3}{2}}}{\frac{4\pi}{3}\alpha^3\frac{c^3}{H_0^3}} = \frac{3\pi}{2\alpha^3}|\Omega_0-1|^{-\frac{3}{2}} \simeq 0.14\,|\Omega_0-1|^{-\frac{3}{2}}$

Planck 2018:
$\underbrace{\Omega_0 = 1 \pm 0.008}_{\text{if } |\Omega_0-1| = 0.008}$
$\Rightarrow \frac{V}{V_{obs}} \simeq 200$ $[\mathcal{O}(100)]$

2) $\tilde{k} \overset{!}{=} -1 \rightarrow$ 3D surface of a 4-hyperboloid

Constraint relation: $u^2 - x^2 - y^2 - z^2 = R_0^2$

Natural coordinates:
$$\left.\begin{aligned} x &= R_0\, \mathrm{sh}\chi \sin\theta\cos\varphi \\ y &= R_0\, \mathrm{sh}\chi \sin\theta \sin\varphi \\ z &= R_0\, \mathrm{sh}\chi \cos\theta \\ u &= R_0\, \mathrm{ch}\chi \end{aligned}\right\} \begin{aligned} d\sigma^2 &= dx^2+dy^2+dz^2-du^2 \\ \tilde{r} &\overset{!}{=} \mathrm{sh}\chi \end{aligned}$$