Friedmann equations

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$$x = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^2 & & \\ & & a^2 & \\ & & & a^2 \end{pmatrix} \Rightarrow g_{\mu\nu} g^{\mu\nu} \overset{!}{=} 4 \Rightarrow g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & \frac{1}{a^2} & & \\ & & \frac{1}{a^2} & \\ & & & \frac{1}{a^2} \end{pmatrix}$$

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + p g^{\mu\nu} \qquad u^\mu = (c, 0, 0, 0)$$

[energy-momentum tensor] [four velocity]

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & & & \\ & \frac{p}{a^2} & & \\ & & \frac{p}{a^2} & \\ & & & \frac{p}{a^2} \end{pmatrix} \qquad T_{\mu\nu} = g_{\mu\beta} g_{\nu\alpha} T^{\alpha\beta} = \begin{pmatrix} \rho c^2 & & & \\ & pa^2 & & \\ & & pa^2 & \\ & & & pa^2 \end{pmatrix}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad \text{[Einstein tensor]}$$

$$R = g_{\mu\nu} R^{\mu\nu} = g^{\mu\nu} R_{\mu\nu} \quad \text{[Ricci scalar]}$$

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} \quad \text{[Ricci tensor]}$$

↳ $\dots_{,\alpha}$ = partial derivation by index „α"

$$\Gamma^\alpha_{\mu\nu,\alpha} = \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha}$$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right) \quad \text{[Christoffel symbol]}$$

$$\boxed{\rightarrow a' = \frac{da}{d(ct)} = \frac{1}{c}\dot{a} \qquad \rightarrow a'' = \frac{d^2a}{d(ct)^2} = \frac{1}{c^2}\ddot{a}}$$

$$\Gamma^0_{ij} = \frac{1}{2}(-1)(-2aa')\delta_{ij} = \boxed{aa'\delta_{ij}}$$

$$\Gamma^i_{0j} = \frac{1}{2}\frac{1}{a^2}\delta^{ik}\cdot 2aa'\cdot\delta_{jk} = \boxed{\frac{a'}{a}\delta^i_j = \Gamma^i_{j0}}$$

$$R_{00} = 0 - \Gamma^\alpha_{0\alpha,0} + 0 - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha} = -3\frac{a''}{a} + 3\frac{a'^2}{a^2} - 3\frac{a'^2}{a^2} = \boxed{-3\frac{a''}{a}}$$

$$R_{ij} = \Gamma^0_{ij,0} - \Gamma^0_{kj}\Gamma^k_{i0} + \Gamma^k_{0k}\Gamma^0_{ij} - \Gamma^k_{0j}\Gamma^0_{ik} = (a'^2 + aa'')\delta_{ij} - a'^2\delta_{ij} + 3a'^2\delta_{ij} - a'^2\delta_{ij} = \boxed{= (aa'' + 2a'^2)\delta_{ij}}$$

$$R = g^{00}R_{00} + g^{ij}R_{ij} = 3\frac{a''}{a} + \frac{1}{a^2}(aa'' + 2a'^2)\cdot 3 = \boxed{6\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)}$$

$$G_{00} = R_{00} - \frac{1}{2}g_{00}R = -3\frac{a''}{a} - \frac{1}{2}(-1)\cdot 6\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right) = \boxed{3\frac{a'^2}{a^2}}$$

$$G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R = (aa'' + 2a'^2)\delta_{ij} - \frac{1}{2}a^2\delta_{ij}\cdot 6\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right) = \boxed{-(a'^2 + 2aa'')\delta_{ij}}$$

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$$G_{\mu\nu} = \begin{pmatrix} 3\frac{a'^2}{a^2} & & & \\ & -(a'^2+2aa'') & & \\ & & -(a'^2+2aa'') & \\ & & & -(a'^2+2aa'') \end{pmatrix}$$

(1) $$G_{00} = \frac{8\pi G}{c^4} T_{00}$$

$$3\frac{a'^2}{a^2} = \frac{8\pi G}{c^4}\varrho c^2$$

$$\frac{3}{c^2}\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{c^2}\varrho$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varrho}$$

With $k \neq 0$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varrho - \frac{kc^2}{a^2}$$

(2) $$g^{\mu\nu}G_{\mu\nu} = \frac{8\pi G}{c^4} g^{\mu\nu}T_{\mu\nu}$$

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu} = \frac{8\pi G}{c^4}\left[\left(\varrho + \frac{p}{c^2}\right)u_\mu u_\nu g^{\mu\nu} + pg_{\mu\nu}g^{\mu\nu}\right]$$

$$R - \frac{1}{2}\cdot R\cdot 4 = \frac{8\pi G}{c^4}\left[\left(\varrho + \frac{p}{c^2}\right)(-c^2) + 4p\right]$$

$$-R = \frac{8\pi G}{c^4}\left[-\varrho c^2 + 3p\right]$$

$$-6\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right) = -\frac{8\pi G}{c^2}\left(\varrho - \frac{3p}{c^2}\right)$$

$$-\frac{6}{c^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = -\frac{8\pi G}{c^2}\left(\varrho - \frac{3p}{c^2}\right)$$

$$\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \frac{4\pi G}{3}\left(\varrho - \frac{3p}{c^2}\right) \quad /(1)$$

$$\frac{\ddot{a}}{a} + \frac{8\pi G}{3}\varrho = \frac{4\pi G}{3}\left(\varrho - \frac{3p}{c^2}\right)$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\varrho + \frac{3p}{c^2}\right)}$$

With $k \neq 0$:

$$R = 6\left(\frac{a''}{a} + \frac{a'^2}{a^2} + \frac{k}{a^2}\right)$$

Thus: $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\varrho + \frac{3p}{c^2}\right)$$

Friedmann equations

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$$x = (x^0, x^1, x^2, x^3) = (ct, r, \theta, \varphi)$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & \frac{a^2}{1-kr^2} & & \\ & & a^2r^2 & \\ & & & a^2r^2\sin^2\theta \end{pmatrix} \Rightarrow g_{\mu\nu}g^{\mu\nu} \overset{!}{=} 4 \Rightarrow g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & \frac{1-kr^2}{a^2} & & \\ & & \frac{1}{a^2r^2} & \\ & & & \frac{1}{a^2r^2\sin^2\theta} \end{pmatrix}$$

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Perfect fluid in comoving frame:

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u^\mu u^\nu + p g^{\mu\nu} \qquad u^\mu = (c, 0, 0, 0)$$

[energy-momentum tensor] [four velocity]

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & & & \\ & \frac{p}{a^2}(1-kr^2) & & \\ & & \frac{p}{a^2}\frac{1}{r^2} & \\ & & & \frac{p}{a^2}\frac{1}{r^2\sin^2\theta} \end{pmatrix} \Rightarrow T_{\mu\nu} = g_{\mu\beta}g_{\nu\alpha}T^{\alpha\beta} = \begin{pmatrix} \rho c^2 & & & \\ & \frac{pa^2}{1-kr^2} & & \\ & & pa^2r^2 & \\ & & & pa^2r^2\sin^2\theta \end{pmatrix}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \quad \text{[Einstein tensor]}$$

$$R = g_{\mu\nu}R^{\mu\nu} = g^{\mu\nu}R_{\mu\nu} \quad \text{[Ricci scalar]}$$

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} \quad \text{[Ricci tensor]} \Rightarrow \Gamma^\alpha_{\mu\nu,\alpha} = \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha}$$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right) \quad \text{[Christoffel symbol]}$$

$$\boxed{\begin{aligned} &\rightarrow \acute{a} = \frac{da}{d(ct)} = \frac{1}{c}\dot{a} \\ &\rightarrow \acute{\acute{a}} = \frac{d^2a}{d(ct)^2} = \frac{1}{c^2}\ddot{a} \end{aligned}}$$

$$\Gamma^\phi_{11} = -\frac{1}{2}g^{\phi\phi}\frac{\partial g_{11}}{\partial x^\phi} = \frac{a\acute{a}}{1-kr^2}$$

$$\Gamma^\phi_{22} = -\frac{1}{2}g^{\phi\phi}\frac{\partial g_{22}}{\partial x^\phi} = a\acute{a}r^2$$

$$\Gamma^\phi_{33} = -\frac{1}{2}g^{\phi\phi}\frac{\partial g_{33}}{\partial x^\phi} = a\acute{a}r^2\sin^2\theta$$

$$\Gamma^1_{\phi 1} = \Gamma^1_{1\phi} = \frac{1}{2}g^{11}\frac{\partial g_{11}}{\partial x^\phi} = \frac{\acute{a}}{a}$$

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\frac{\partial g_{11}}{\partial x^1} = \frac{kr}{1-kr^2}$$

$$\Gamma^1_{22} = -\frac{1}{2}g^{11}\frac{\partial g_{22}}{\partial x^1} = -r(1-kr^2)$$

$$\Gamma^1_{33} = -\frac{1}{2}g^{11}\frac{\partial g_{33}}{\partial x^1} = -r\sin^2\theta(1-kr^2)$$

$$\Gamma^2_{\phi 2} = \Gamma^2_{2\phi} = \frac{1}{2}g^{22}\frac{\partial g_{22}}{\partial x^\phi} = \frac{\acute{a}}{a}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}\frac{\partial g_{22}}{\partial x^1} = \frac{1}{r}$$

$$\Gamma^2_{33} = -\frac{1}{2}g^{22}\frac{\partial g_{33}}{\partial x^2} = -\sin\theta\cos\theta$$

$$\Gamma^3_{\phi 3} = \Gamma^3_{3\phi} = \frac{1}{2}g^{33}\frac{\partial g_{33}}{\partial x^\phi} = \frac{\acute{a}}{a}$$

$$\Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{2}g^{33}\frac{\partial g_{33}}{\partial x^1} = \frac{1}{r}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2}g^{33}\frac{\partial g_{33}}{\partial x^2} = \frac{\cos\theta}{\sin\theta}$$

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$\boxed{\text{Friedmann equations}}$

$$R_{\phi\phi} = -\Gamma^{i}_{\phi i,\phi} - \left(\Gamma^{1}_{1\phi}\right)^2 - \left(\Gamma^{2}_{2\phi}\right)^2 - \left(\Gamma^{3}_{3\phi}\right)^2 = -3\frac{a''}{a}$$

$$R_{\phi 1} = R_{1\phi} = R_{\phi 2} = R_{2\phi} = R_{\phi 3} = R_{3\phi} = \phi$$

$$R_{11} = \Gamma^{\phi}_{11,\phi} - \Gamma^{2}_{12,1} - \Gamma^{3}_{13,1} + \Gamma^{1}_{\phi 1}\Gamma^{\phi}_{11} + \Gamma^{2}_{12}\Gamma^{1}_{11} + \Gamma^{3}_{13}\Gamma^{1}_{11} - \left(\Gamma^{2}_{21}\right)^2 - \left(\Gamma^{3}_{31}\right)^2 = \frac{1}{1-kr^2}\left(aa'' + 2a'^2\right) + \frac{2k}{1-kr^2}$$

$$R_{12} = R_{21} = R_{13} = R_{31} = \phi$$

$$R_{22} = \Gamma^{\phi}_{22,\phi} + \Gamma^{1}_{22,1} - \Gamma^{3}_{23,2} + \Gamma^{1}_{\phi 1}\Gamma^{\phi}_{22} + \Gamma^{1}_{11}\Gamma^{1}_{22} - \Gamma^{3}_{32}\Gamma^{3}_{32} = r^2\left(aa'' + 2a'^2\right) + 2kr^2$$

$$R_{23} = R_{32} = \phi$$

$$R_{33} = \Gamma^{\phi}_{33,\phi} + \Gamma^{1}_{33,1} + \Gamma^{2}_{33,2} + 2\Gamma^{1}_{\phi 1}\Gamma^{\phi}_{33} + \Gamma^{1}_{11}\Gamma^{1}_{33} - \Gamma^{2}_{33}\Gamma^{3}_{32} = r^2\sin^2\theta\left(aa'' + 2a'^2\right) + 2kr^2\sin^2\theta$$

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\phi\phi}R_{\phi\phi} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} = \frac{6}{c^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right)$$

$\boxed{\text{FI}}$

$$G_{\phi\phi} = R_{\phi\phi} - \frac{1}{2}Rg_{\phi\phi} = \frac{3}{c^2}\left(\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right) \overset{!}{=} \frac{8\pi G}{c^4}\rho c^2$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}}$$

$\boxed{\text{FII}}$

$$g^{\mu\nu}G_{\mu\nu} = \underbrace{g^{\mu\nu}R_{\mu\nu}}_{R} - \frac{1}{2}R\underbrace{g^{\mu\nu}g_{\mu\nu}}_{4} = -R \overset{!}{=} \frac{8\pi G}{c^4}g^{\mu\nu}T_{\mu\nu}$$

$$\cancel{-}\frac{6}{\cancel{c^2}}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right) = \cancel{-}\frac{8\pi G}{\cancel{c^2}}\left(\rho - \frac{3p}{c^2}\right) \qquad \leftarrow (\text{FI})$$

$$\frac{\ddot{a}}{a} + \frac{8\pi G}{3}\rho - \cancel{\frac{kc^2}{a^2}} + \cancel{\frac{kc^2}{a^2}} = \frac{4\pi G}{3}\left(\rho - \frac{3p}{c^2}\right)$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)}$$

Energy-momentum conservation

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$$T^{\mu\nu}{}_{;\nu} \overset{!}{=} 0$$

$$T^{\mu\nu}{}_{;\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^{\mu}_{\nu\alpha} T^{\alpha\nu} + \Gamma^{\nu}_{\nu\alpha} T^{\mu\alpha}$$

1-2-3: $T^{i\nu}{}_{;\nu} = 0 \overset{!}{=} 0$ ✓

0:

$$T^{0\nu}{}_{;\nu} = \frac{\partial T^{00}}{\partial x^0} + \Gamma^{0}_{11} T^{11} + \Gamma^{0}_{22} T^{22} + \Gamma^{0}_{33} T^{33} + \Gamma^{i}_{i0} T^{00}$$

$$\Rightarrow \quad \dot{\varrho} c^2 + \frac{3\dot{a}}{a}(\varrho c^2 + p) \overset{!}{=} 0$$

$$\dot{\varrho} + \frac{3\dot{a}}{a}\left(\varrho + \frac{p}{c^2}\right) = 0$$

$$\frac{\dot{\varrho}}{\dot{a}} = -\frac{3}{a}\left(\varrho + \frac{p}{c^2}\right) \qquad \leftarrow p = w\varrho c^2$$

$$\frac{d\varrho}{da} = -\frac{3}{a}(1+w)\varrho$$

$$\frac{d\varrho}{\varrho} = -3(1+w)\frac{da}{a}$$

$$d(\ln\varrho) = d\left(\ln a^{-3(1+w)}\right)$$

$$\ln\varrho = \ln a^{-3(1+w)} + \ln\varrho_0$$

$$\varrho = \varrho_0 a^{-3(1+w)}$$

$$\boxed{\varrho(a) = \varrho_0 a^{-3(1+w)}}$$