Solving Friedmann's equations

(1) $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$

(2) $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$

(3) $p = w\rho c^2$

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(1) $\rightarrow$ $H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$ $\quad / : H^2$

$1 = \frac{8\pi G}{3H^2}\rho - \frac{kc^2}{a^2H^2}$ $\qquad \rho_c = \frac{3H^2}{8\pi G}$; $\rho_{c,0} = \frac{3H_0^2}{8\pi G}$

$1 = \frac{\rho}{\rho_c} - \frac{kc^2}{a^2H^2}$ $\qquad \Omega = \frac{\rho}{\rho_c}$; $\Omega_0 = \frac{\rho_0}{\rho_{c,0}}$

$\Omega_k = -\frac{kc^2}{a^2H^2}\left(= -\frac{kc^2}{\dot{a}^2}\right)$; $\Omega_{k,0} = -\frac{kc^2}{H_0^2}$

$\hookrightarrow \frac{\Omega_k}{\Omega_{k,0}} = -\frac{kc^2}{a^2H^2} \cdot -\frac{H_0^2}{kc^2} = \underbrace{\frac{H_0^2}{H^2}\frac{1}{a^2}}$

$\boxed{\Omega_k = \frac{H_0^2}{H^2}\Omega_{k,0}\, a^{-2}}$

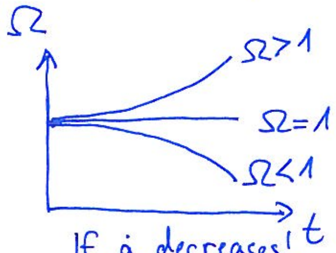
$1 = \Omega + \Omega_k$

$\Omega_k = 1 - \Omega$

$\Omega_{k,0} = 1 - \Omega_0$

$\longrightarrow$
$\Omega > 1 \Rightarrow \Omega_k < 0 \Rightarrow k > 0$
$\Omega < 1 \Rightarrow \Omega_k > 0 \Rightarrow k < 0$
$\Omega = 1 \Rightarrow \Omega_k = 0 \Rightarrow k = 0$

$\frac{H_0^2\Omega_{k,0}}{a^2H^2} = \boxed{\frac{H_0^2\Omega_{k,0}}{\dot{a}^2} = 1 - \Omega}$



If $\dot{a}$ decreases!

$\rightarrow$ $\rho = \frac{m}{V} = \frac{m_m + m_r + m_\Lambda}{V} = \rho_m + \rho_r + \rho_\Lambda$

$\Omega = \frac{\rho}{\rho_c} = \frac{\rho_m}{\rho_c} + \frac{\rho_r}{\rho_c} + \frac{\rho_\Lambda}{\rho_c} = \Omega_m + \Omega_r + \Omega_\Lambda$

$\left.\begin{array}{l} \rho(a) = \rho_0\, a^{-3(1+w)} \\ \rho_c = \rho_{c,0} \cdot \frac{H^2}{H_0^2} \end{array}\right\}$ $\Omega_x = \frac{\rho_x}{\rho_c} = \frac{\rho_{x,0}\, a^{-3(1+w_x)}}{\rho_{c,0}\ \frac{H^2}{H_0^2}} = \frac{H_0^2}{H^2}\Omega_{x,0}\, a^{-3(1+w_x)}$

$\Omega_m = \frac{H_0^2}{H^2}\Omega_{m,0}\, a^{-3}$ $\qquad \Omega_\Lambda = \frac{H_0^2}{H^2}\Omega_{\Lambda,0}\, a^{0}$

$\Omega_r = \frac{H_0^2}{H^2}\Omega_{r,0}\, a^{-4}$ $\qquad \hookrightarrow \Omega_k = \frac{H_0^2}{H^2}\Omega_{k,0}\, a^{-2} \Rightarrow w_k = -\frac{1}{3}$

$$\left.\begin{array}{l}\Omega_{\Lambda,0} \simeq 0.7 \\ \Omega_{m,0} \simeq 0.3 \\ \Omega_{r,0} \simeq 10^{-4}\end{array}\right\} \Omega_0 = \Omega_{m,0} + \Omega_{r,0} + \Omega_{\Lambda,0} \simeq 1$$

$$\Omega_{k,0} \simeq \phi$$

$$\frac{\Omega_m}{\Omega} = \frac{\frac{H_0^{\cancel{2}}}{\cancel{H^2}} \Omega_{m,0} a^{-3}}{\frac{H_0^{\cancel{2}}}{\cancel{H^2}} (\Omega_{m,0} a^{-3} + \Omega_{r,0} a^{-4} + \Omega_{\Lambda,0})}$$

→ At what „a" was $\Omega_m = \Omega_\Lambda$?

$$\frac{H_0^{\cancel{2}}}{\cancel{H^2}} \Omega_{m,0} a^{-3} = \frac{H_0^{\cancel{2}}}{\cancel{H^2}} \Omega_{\Lambda,0}$$

$$a = \sqrt[3]{\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}} \simeq 0.75 \quad (0.77)$$

→ At what „a" was $\Omega_r = 0.5$?

$$\Omega_r = \frac{\Omega_0}{2} = \frac{\Omega}{2}$$

$$\frac{H_0^{\cancel{2}}}{\cancel{H^2}} \Omega_{r,0} a^{-4} = \frac{1}{2} \frac{H_0^{\cancel{2}}}{\cancel{H^2}} (\Omega_{m,0} a^{-3} + \Omega_{r,0} a^{-4} + \underbrace{\Omega_{\Lambda,0}}_{\simeq \phi})$$

$$\Omega_{r,0} = \frac{1}{2} (\Omega_{m,0} a + \Omega_{r,0})$$

$$\Omega_{r,0} = \Omega_{m,0} a \qquad (\Omega_r = \Omega_m)$$

$$a = \frac{\Omega_{r,0}}{\Omega_{m,0}} \simeq 3 \cdot 10^{-4} \qquad (3 \cdot 10^{-4})$$

$$1 = \Omega + \Omega_k$$

$$1 = \frac{H_0^2}{H^2}\left(\Omega_{m,0}\, a^{-3} + \Omega_{r,0}\, a^{-4} + \Omega_{\Lambda,0} + \Omega_{k,0}\, a^{-2}\right) \qquad /\cdot H^2$$

$$\frac{\dot{a}^2}{a^2} = H_0^2 \underbrace{\left(\Omega_{m,0}\, a^{-3} + \Omega_{r,0}\, a^{-4} + \Omega_{\Lambda,0} + \Omega_{k,0}\, a^{-2}\right)}_{E^2(a;\Omega\ldots) \to \text{expansion function}}$$

$$\frac{da}{dt} = \pm H_0 \sqrt{\Omega_{m,0}\, a^{-1} + \Omega_{r,0}\, a^{-2} + \Omega_{\Lambda,0}\, a^{2} + \Omega_{k,0}}$$

$\hookrightarrow a\, E(a;\Omega\ldots)$

→ „+" sign: expansion
„−" sign: contraction } symmetric solutions

Let's keep the expanding solution!: → „t" goes from $t=\emptyset$ to $t=T$
„a" goes from $a=\emptyset$ to $a=a_0=1$

$$H_0\, dt = \frac{da}{a\, E(a)}$$

$$H_0 \int_{t_1}^{t_2} dt = \int_{a_1}^{a_2} \frac{da}{a\, E(a)}$$

$$H_0 (t_2 - t_1) = \int_{a_1}^{a_2} \frac{da}{a\, E(a)}$$

For example : $t_1 = \emptyset \iff a_1 = \emptyset$

$$\underbrace{H_0 t = \int_{\emptyset}^{a} \frac{da'}{a'\, E(a')}}_{} \implies t(a) = \frac{1}{H_0}\int_{\emptyset}^{a} \frac{da'}{a'\, E(a')} \implies a(t) = \ldots$$

$$\underbrace{T = \frac{1}{H_0}\int_{\emptyset}^{1} \frac{da}{a\, E(a)}}_{}$$

a_1: λ_0

a_2: λ

$$\lambda = \frac{a_2}{a_1}\lambda_0 \;!$$

If $a_2 = a_0 = 1 \Rightarrow \lambda = \frac{1}{a}\lambda_0$

$$z = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{\frac{1}{a}\lambda_0 - \lambda_0}{\lambda_0} = \frac{1}{a} - 1$$

$$\boxed{a = \frac{1}{1+z}}$$

$$da = -\frac{1}{(1+z)^2}\,dz \qquad \left(dz = -\frac{da}{a^2} \right)$$

With z-s:

$$H_0(t_2 - t_1) = \int_{z_1}^{z_2} -\frac{1}{(1+z)^2}\,dz\,\cancel{(1+z)}\cdot\frac{1}{\underbrace{\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2}}_{E(z;\Omega\ldots)}}$$

$$H_0(t_2 - t_1) = \int_{z_2}^{z_1} \frac{dz}{(1+z)\,E(z)}$$

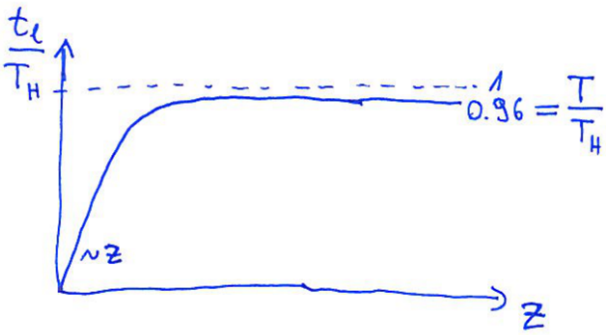
$$H_0 t = \int_{z}^{+\infty} \frac{dz'}{(1+z')\,E(z')} \qquad T = \frac{1}{H_0}\int_{\emptyset}^{+\infty} \frac{dz'}{(1+z')\,E(z')}$$

$$t_{lookback} = T - t = \frac{1}{H_0}\int_{\emptyset}^{+\infty}\ldots - \frac{1}{H_0}\int_{z}^{+\infty}\ldots = \frac{1}{H_0}\int_{\emptyset}^{z}\frac{dz'}{(1+z')E(z')} \longrightarrow \text{lookback time}$$

$$t_{lookback} = \frac{1}{H_0}\int_{\emptyset}^{z} \frac{dz'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{r,0}(1+z')^4 + \Omega_{\Lambda,0} + \Omega_{k,0}(1+z')^2}}$$

$$T_H = \frac{1}{H_0}$$

$$z \approx \emptyset \rightarrow \quad \frac{t_l}{T_H} \approx \int_{\emptyset}^{z} dz' = z$$



$$H_0 t = \int_{\phi}^{a} \frac{da'}{\sqrt{\Omega_{m,0}\, a'^{-1} + \Omega_{r,0}\, a'^{-2} + \Omega_{\Lambda,0}\, a'^{2} + (1 - \Omega_{m,0} - \Omega_{r,0} - \Omega_{\Lambda,0})}}$$

Single-component universe:

$$H_0 t = \int_{\phi}^{a} \frac{da'}{\sqrt{\Omega_{x,0}\, a'^{-3(1+w_x)+2} + 1 - \Omega_{x,0}}}$$

With $\Omega_{k,0} = \phi \Rightarrow \Omega_{x,0} = 1$

$$H_0 t = \int_{\phi}^{a} \frac{da'}{a'^{-\frac{3}{2}(1+w_x)+1}} = \int_{\phi}^{a} \frac{da'}{a'^{-\frac{(1+3w_x)}{2}}} = \int_{\phi}^{a} a'^{\frac{1+3w_x}{2}}\, da'$$

If $w_x \neq -1$:

$$H_0 t = \frac{a^{\frac{1+3w_x}{2}+1}}{\frac{1+3w_x}{2}+1} = \frac{a^{\frac{3(1+w_x)}{2}}}{\frac{3(1+w_x)}{2}}$$

$$a(t) = \left[\frac{3(1+w_x)}{2} H_0\right]^{\frac{2}{3(1+w_x)}} t^{\frac{2}{3(1+w_x)}}$$

Radiation-dominated universe:

$$a(t) = \sqrt{2H_0}\, t^{\frac{1}{2}} \qquad a(t) \sim t^{\frac{1}{2}} \qquad \dot{a}(t) \sim \frac{1}{2} t^{-\frac{1}{2}} \qquad H = \frac{\dot{a}}{a} = \frac{1}{2}\frac{1}{t}$$

Matter-dominated universe:

$$a(t) = \left(\frac{3}{2} H_0\right)^{\frac{2}{3}} t^{\frac{2}{3}} \qquad a(t) \sim t^{\frac{2}{3}} \qquad \dot{a}(t) \sim \frac{2}{3} t^{-\frac{1}{3}} \qquad H = \frac{\dot{a}}{a} = \frac{2}{3}\frac{1}{t}$$

Lambda-dominated universe:

$$H_0 t = \int_{a_i}^{a} a'^{-1}\, da' = \ln a - \ln a_i$$

$$a(t) = a_i\, e^{H_0 t} \qquad \dot{a}(t) = a_i H_0\, e^{H_0 t} \qquad H = \frac{\dot{a}}{a} = H_0 = \text{const.}$$

$\Omega_0 \simeq 1 \quad (\Omega_{k,0} \simeq 0)$

$\Omega_{\Lambda,0} \simeq 0.7$

$\Omega_{m,0} \simeq 0.3$

$\Omega_{r,0} \simeq 10^{-4}$

$H_0 \simeq 70 \frac{km}{s \cdot Mpc}$

$T \simeq 13.8$ Byr