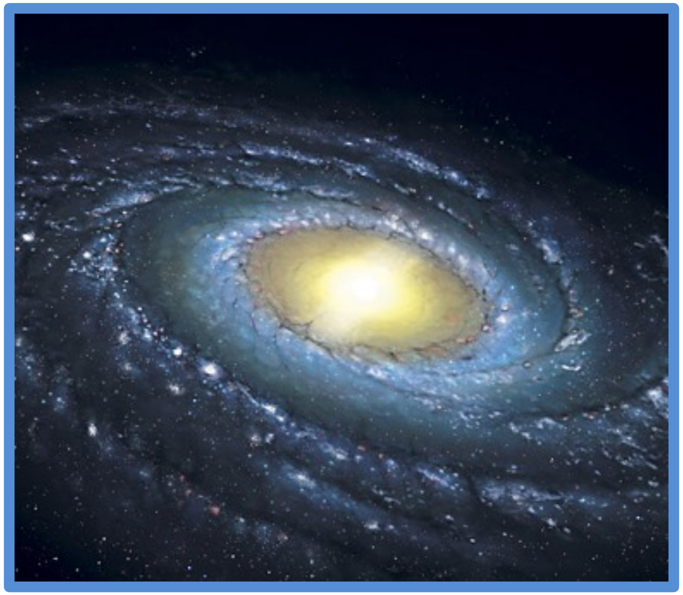
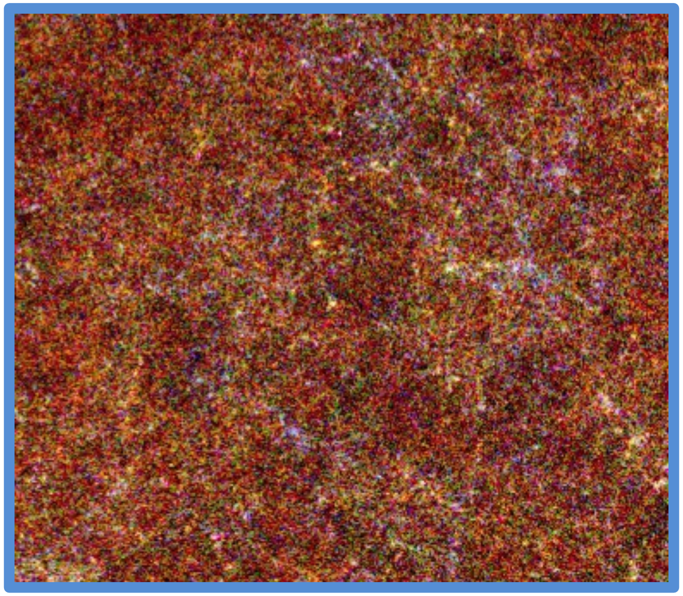
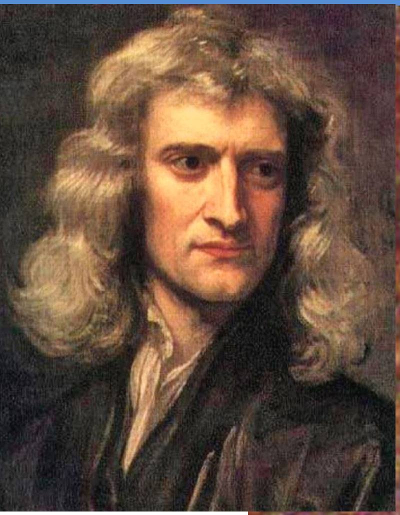
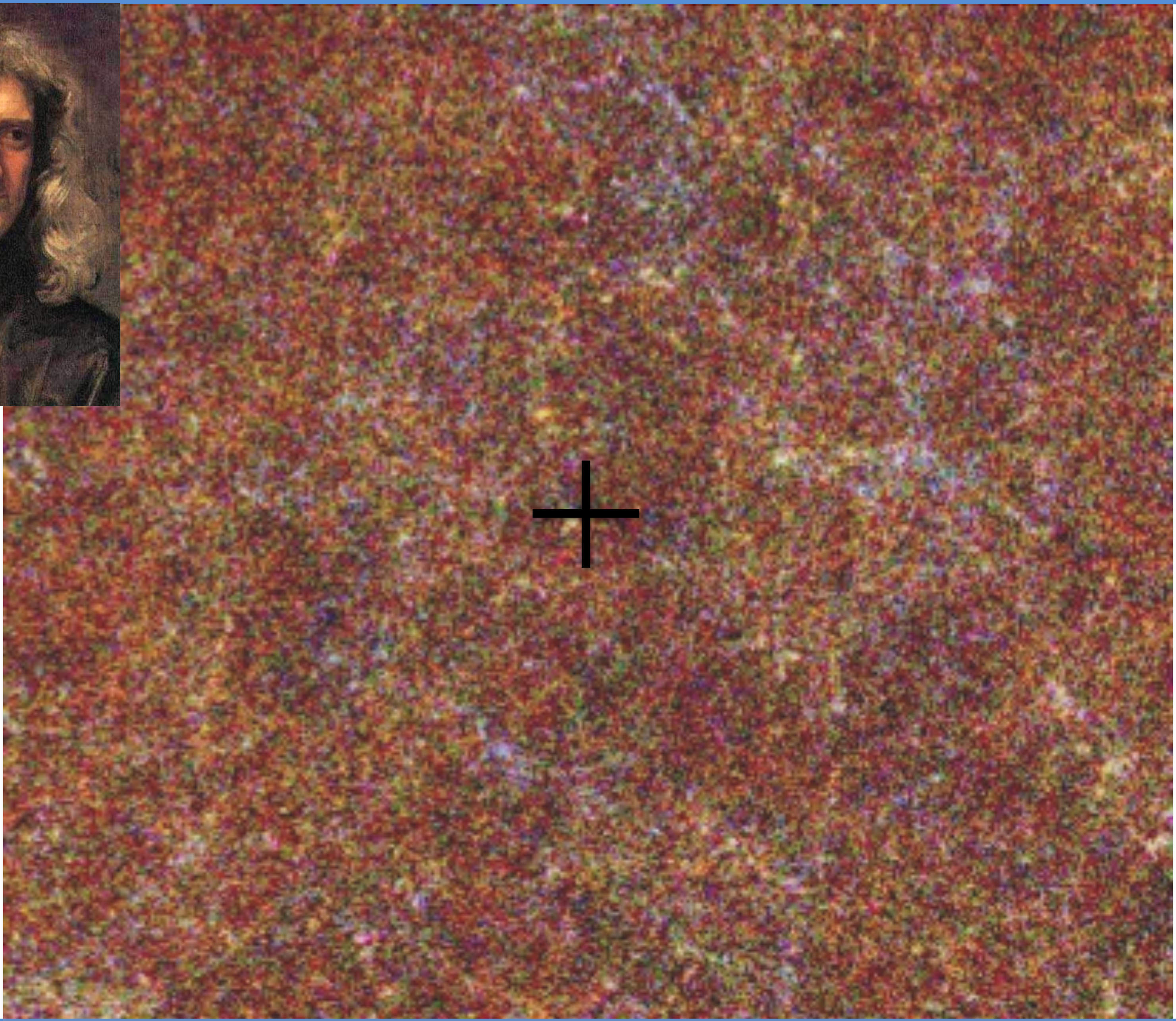
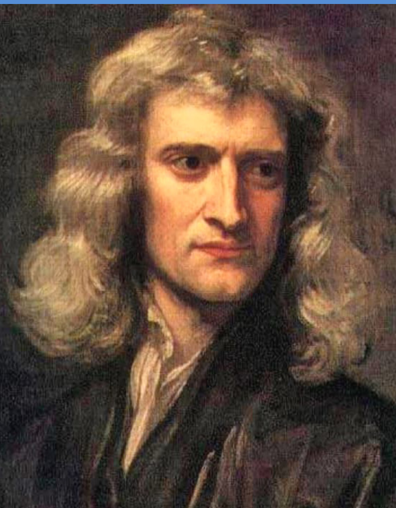
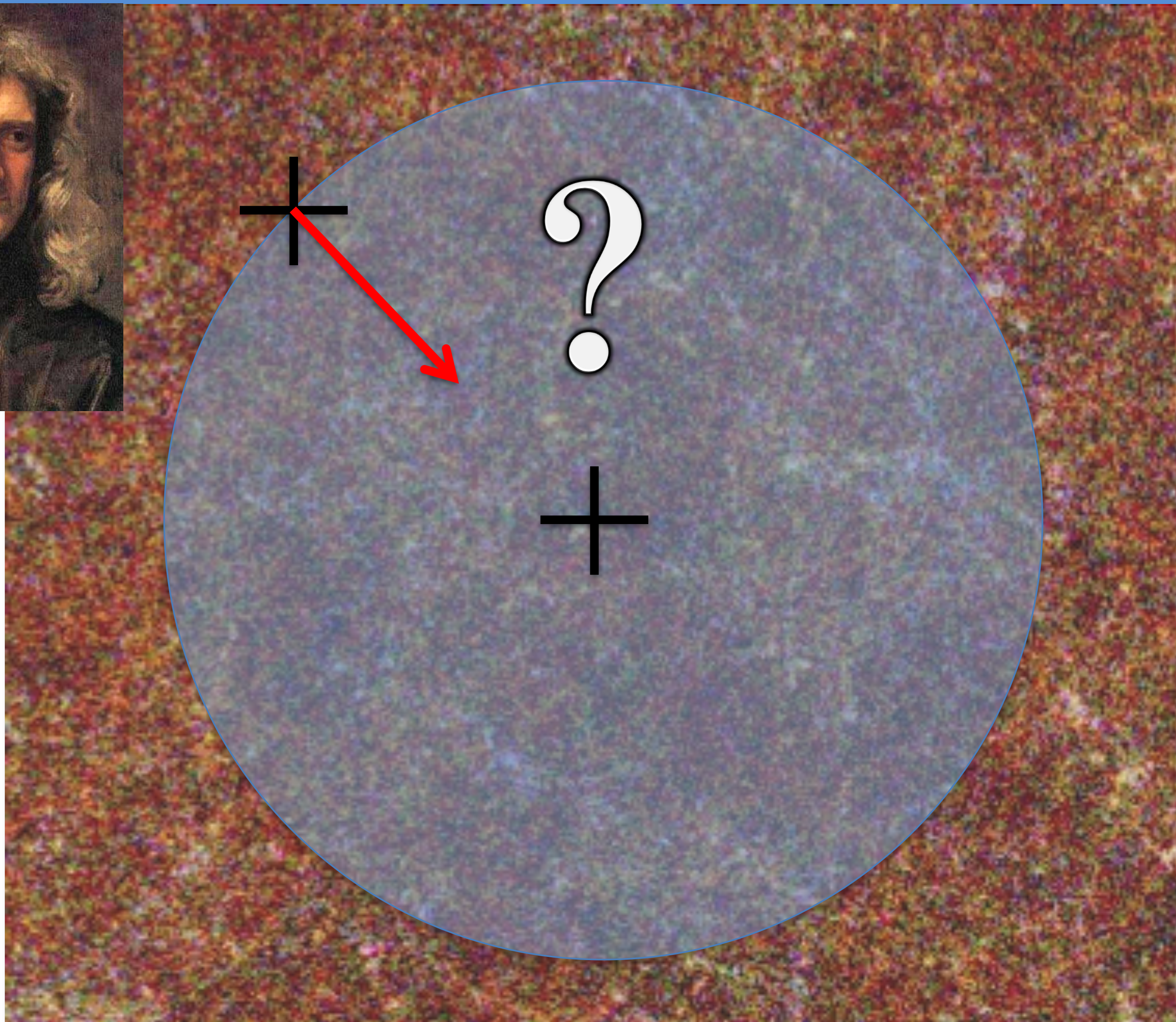









Force on a point inside a hollow sphere [edit]

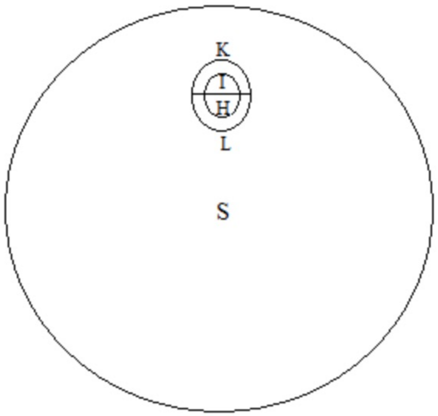


Fig. 1

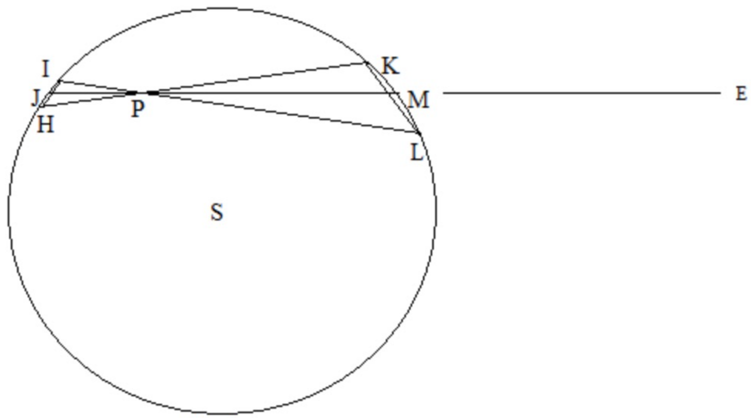


Fig. 2

Fig. 2 is a cross-section of the hollow sphere through the centre, S and an arbitrary point, P, inside the sphere. Through P draw two lines IL and HK such that the angle KPL is very small. JM is the line through P that bisects that angle. From the geometry of circles, the triangles IPH and KPL are similar. The lines KH and IL are rotated about the axis JM to form 2 cones that intersect the sphere in 2 closed curves. In Fig. 1 the sphere is seen from a distance along the line PE and is assumed transparent so both curves can be seen.

The surface of the sphere that the cones intersect can be considered to be flat, and $\angle PJI = \angle PMK$.

Since the intersection of a cone with a plane is an ellipse, in this case the intersections form two ellipses with major axes IH and KL, where $\frac{IH}{KL} = \frac{PJ}{PM}$.

By a similar argument, the minor axes are in the same ratio. This is clear if the sphere is viewed from above. Therefore the two ellipses are similar, so their areas are as the squares of their major axes. As the mass of any section of the surface is proportional to the area of that section, for the 2 elliptical areas the ratios of their masses $\propto \frac{PJ^2}{PM^2}$.

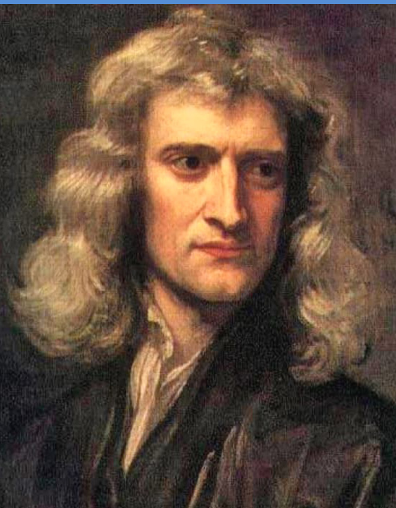
Since the force of attraction on P in the direction JM from either of the elliptic areas, is direct as the mass of the area and inversely as the square of its distance from P, it is independent of the distance of P from the sphere. Hence, the forces on P from the 2 infinitesimal elliptical areas are equal and opposite and there is no net force in the direction JM.

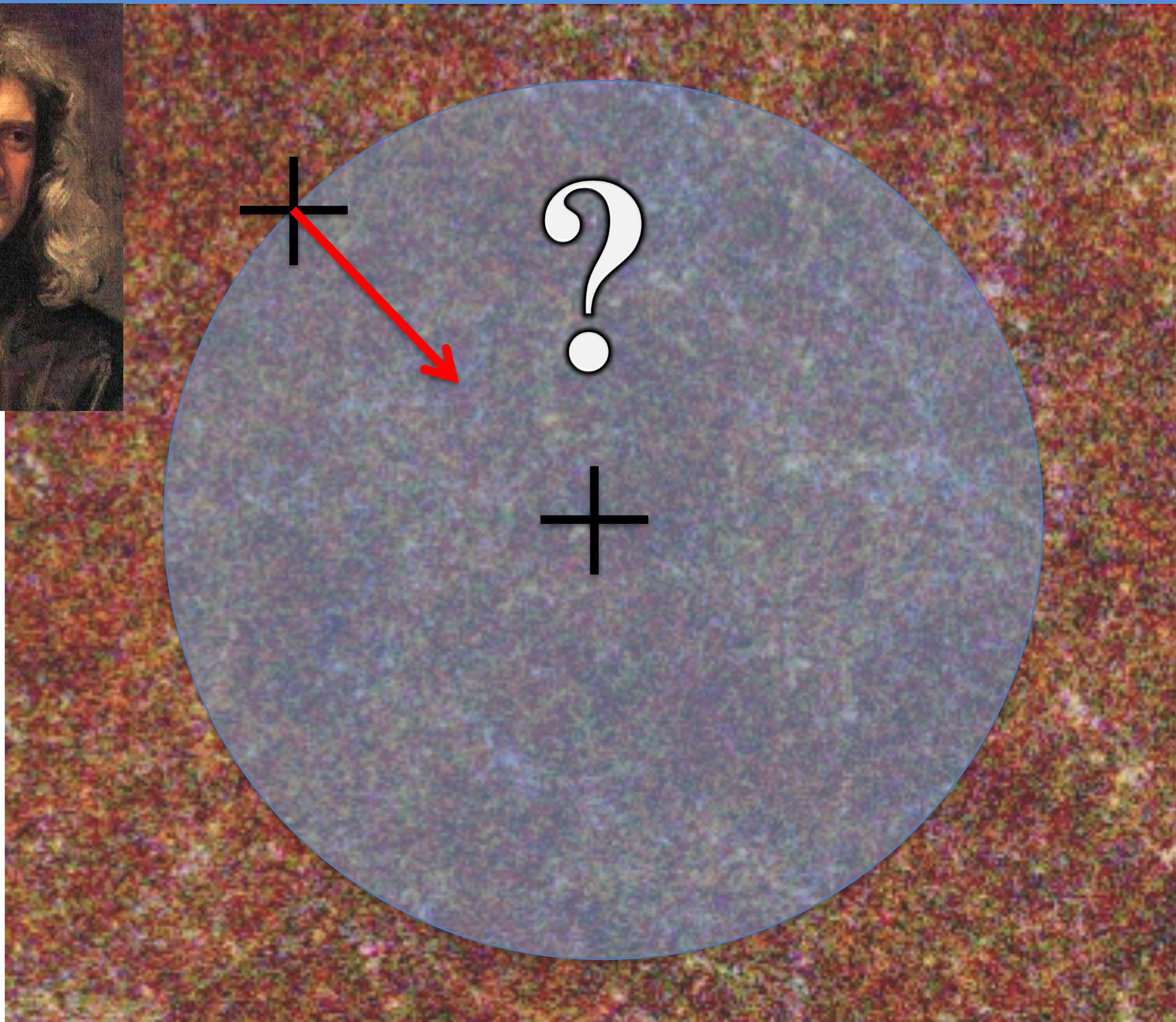
As the position of P and the direction of JM are both arbitrary, it follows that any particle inside a hollow sphere experiences no net force from the mass of the sphere.

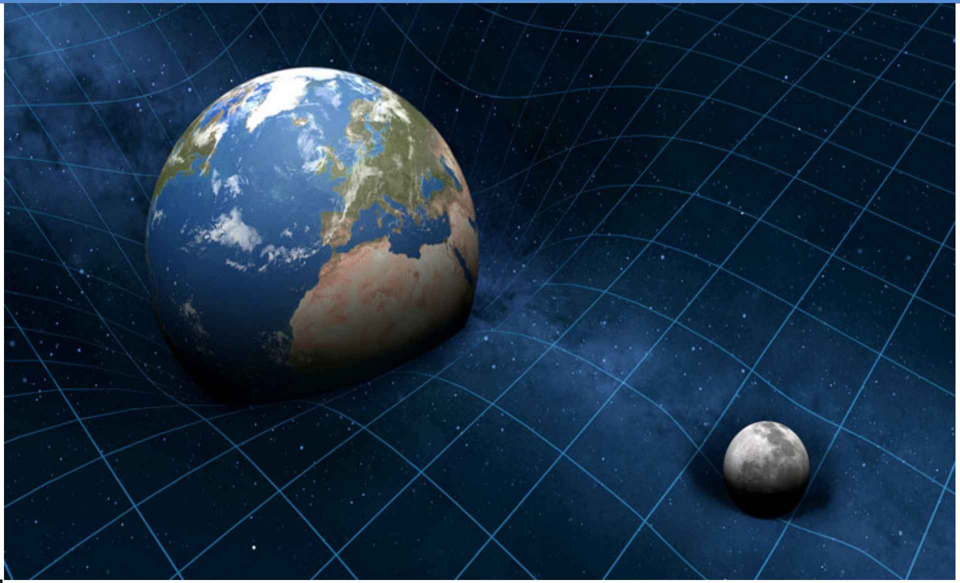
Note: Newton simply describes the arcs IH and KL as 'minimally small' and the areas traced out by the lines IL and HK can be any shape, not necessarily elliptic, but they will always be similar.

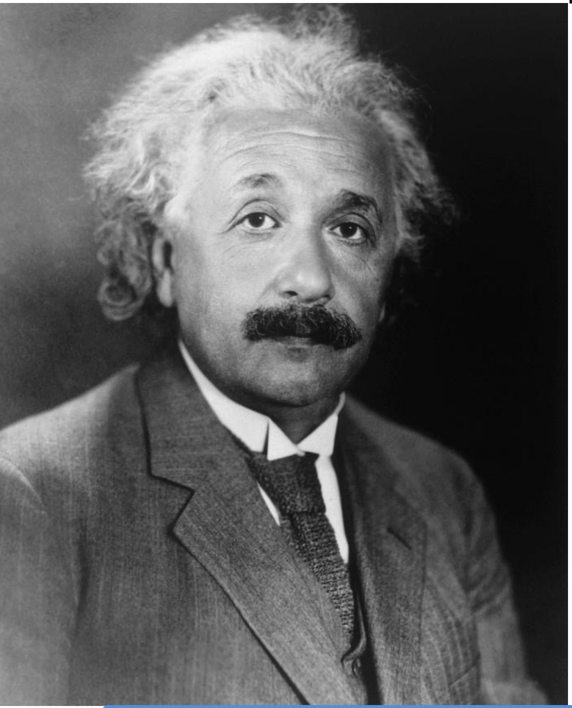
Source: Wikipedia, “Shell theorem” article

https://en.wikipedia.org/wiki/Shell_theorem#Force_on_a_point_inside_a_hollow_sphere

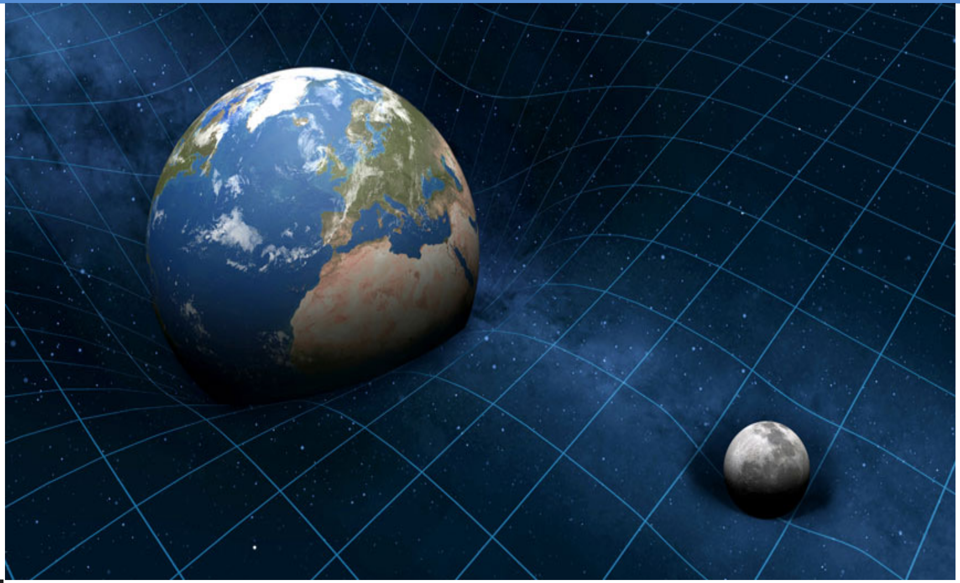


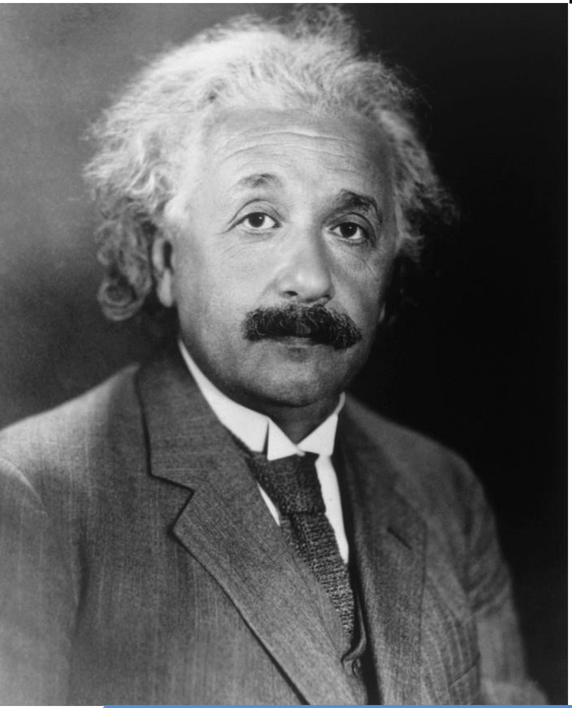






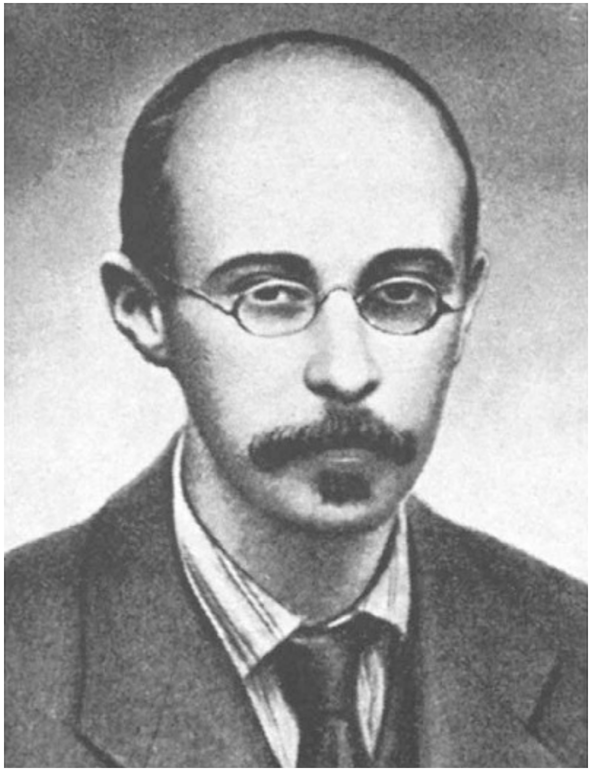
Albert Einstein (1915)





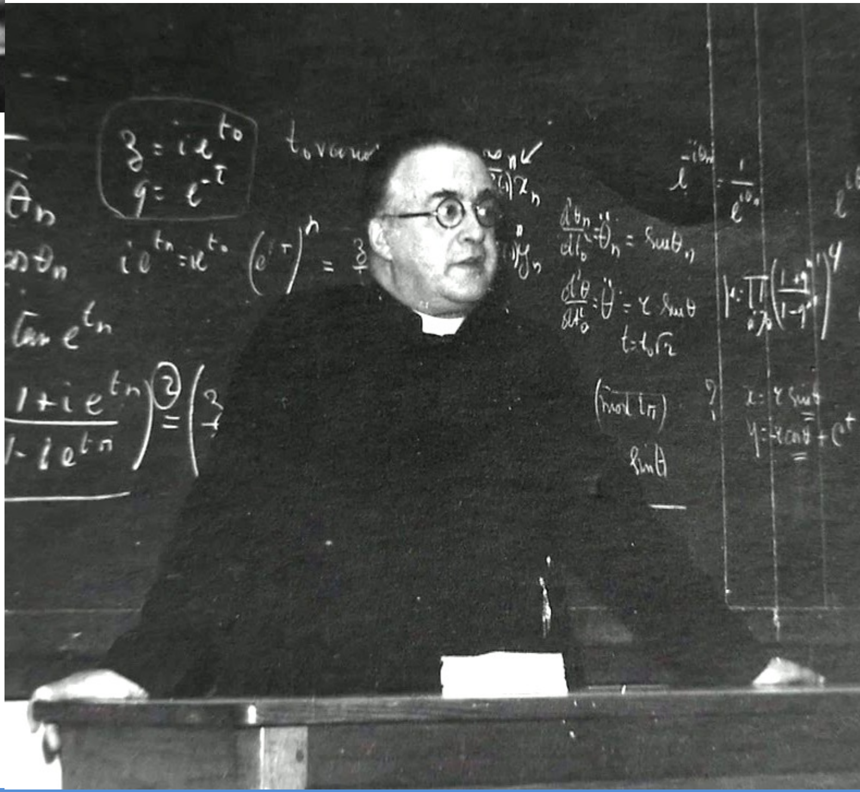
Albert Einstein (1915)

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

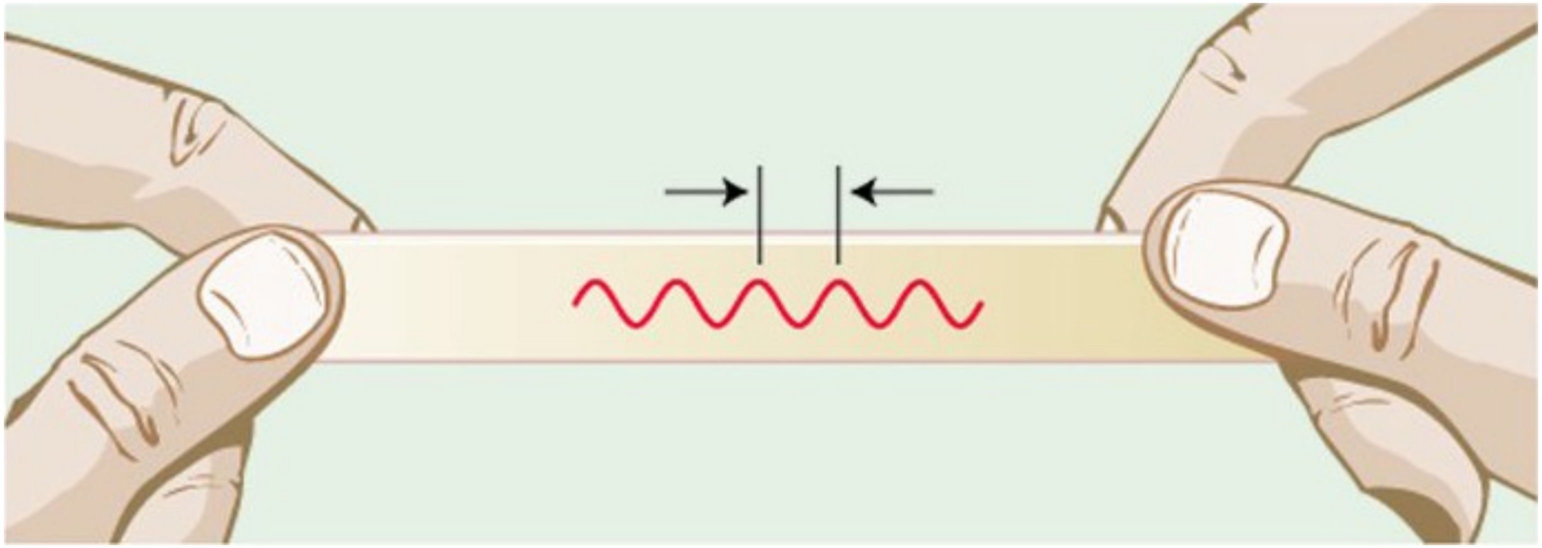


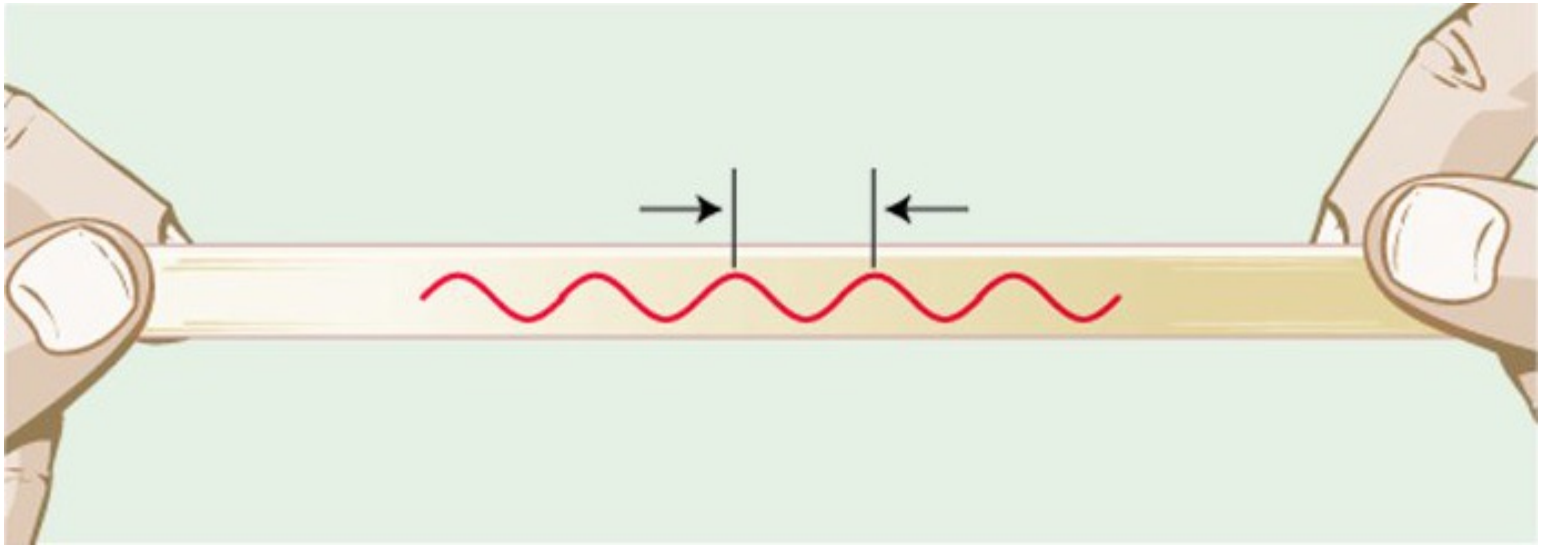
Alexander Friedmann

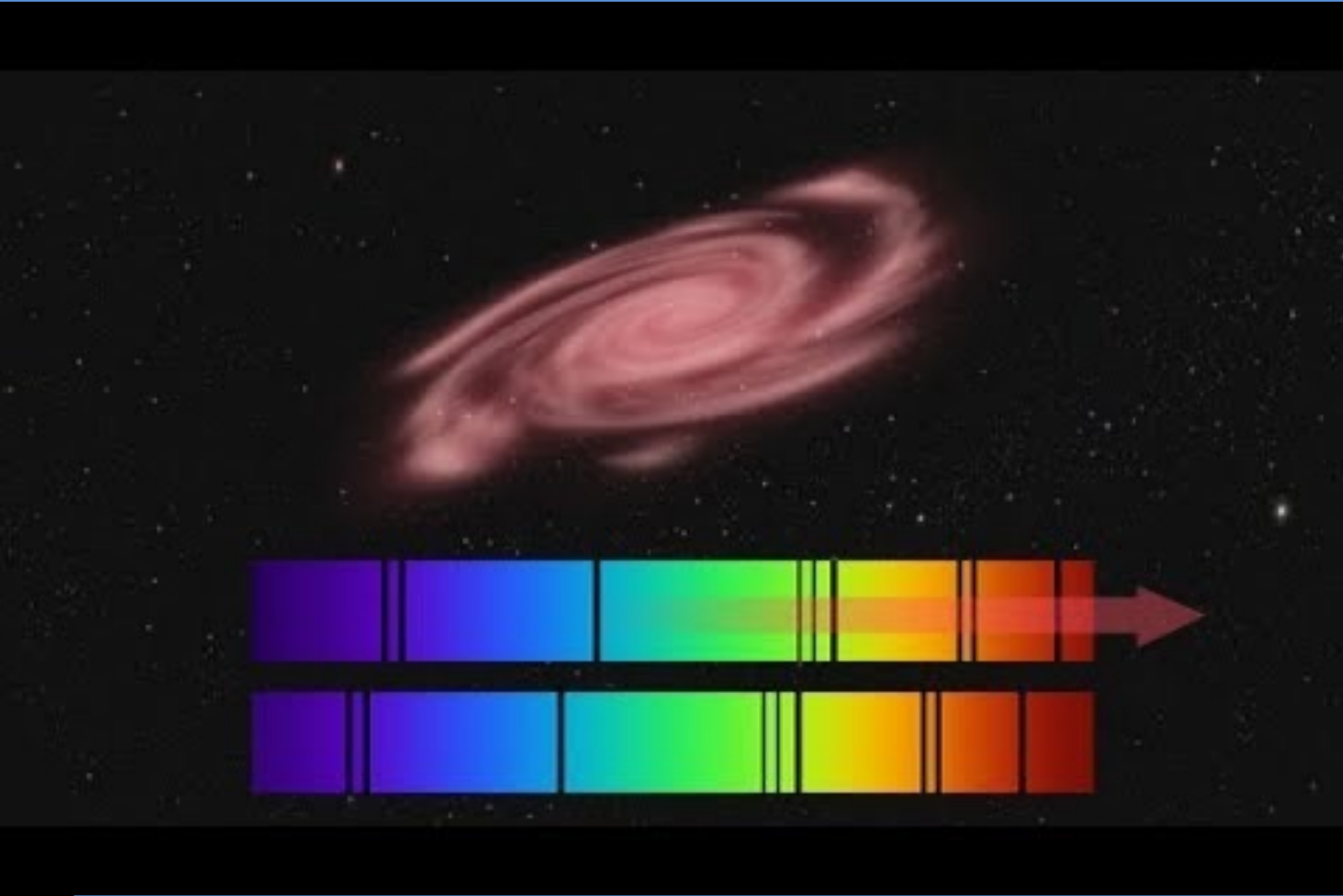
Georges Lemaître



Georges Lemaître
(1894-1966)

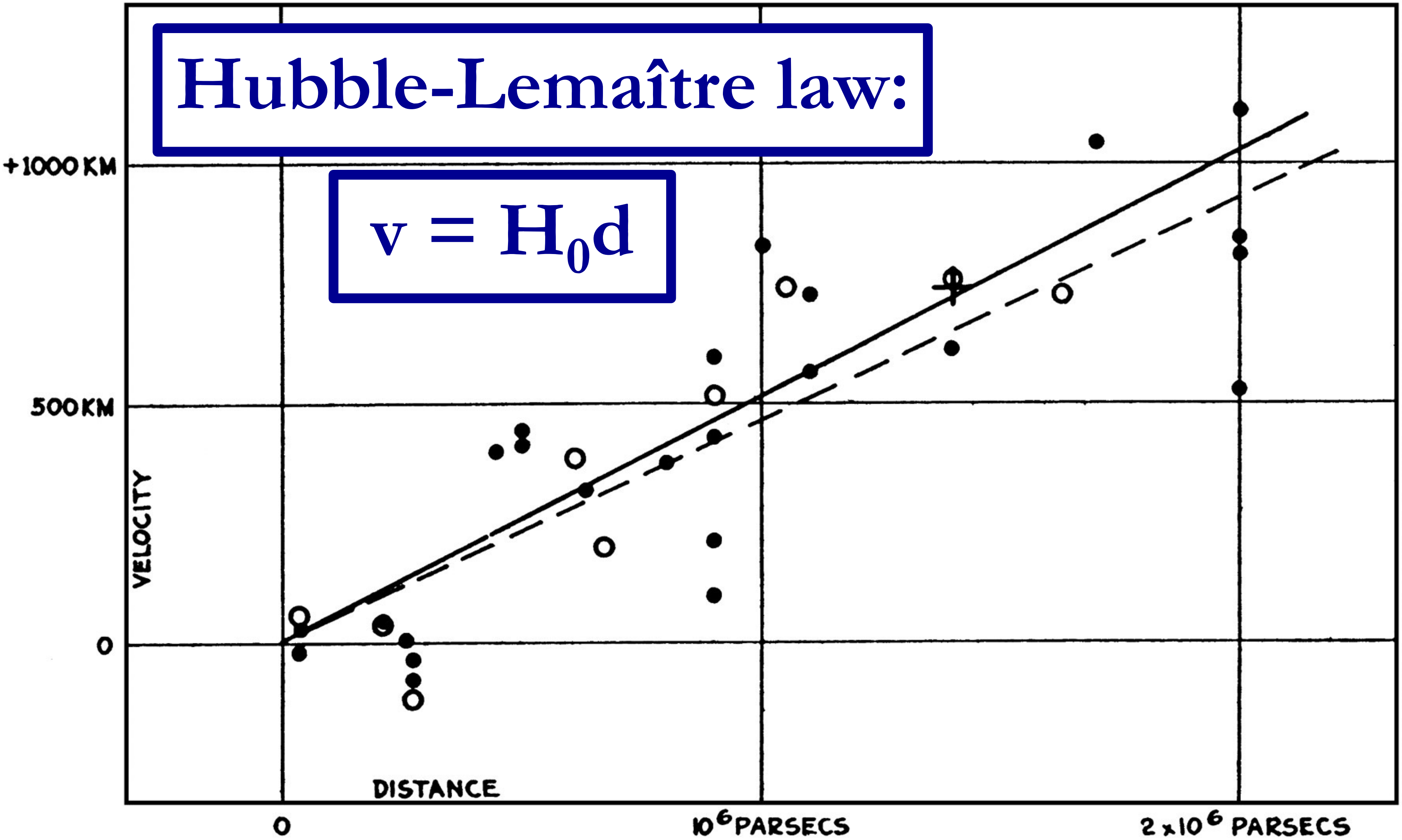


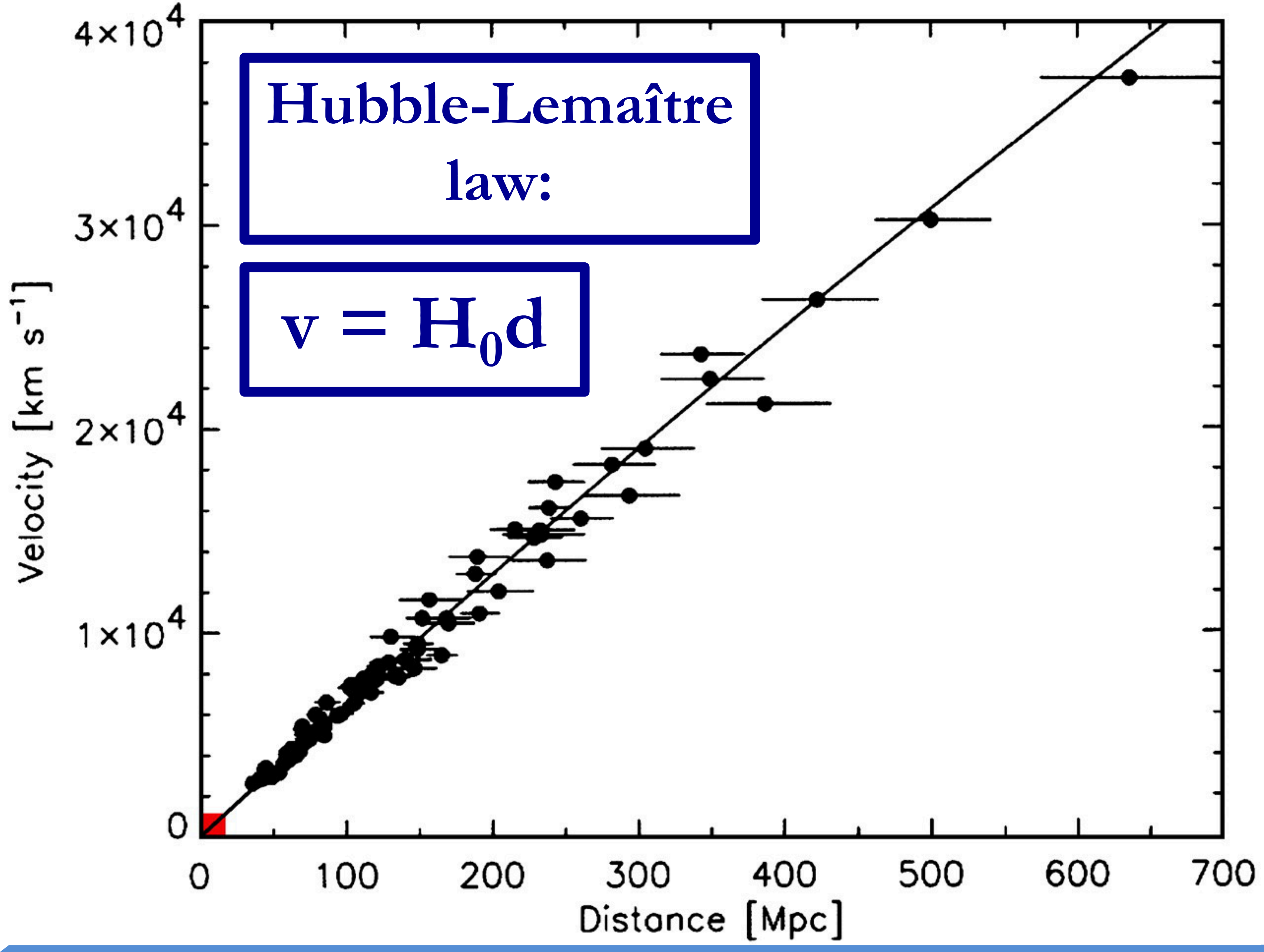




Hubble-Lemaître law:

$$v = H_0 d$$




Hubble-Lemaître law:
v = H0d
Velocity [km s-1]
Distance [Mpc]
4×10^4
3×10^4
2×10^4
1×10^4
0
0
100
200
300
400
500
600
700

Newtonian derivation of the Friedmann equations

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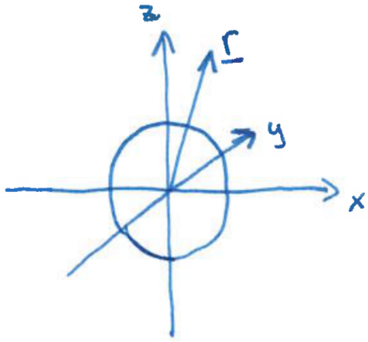

From the cosmological principle:

$$\varrho \overset{!}{=} \varrho(t)$$

Notations: $r = |\underline{r}|$ $\quad \underline{r} = \underline{r}(t)$

$\dot{r} = |\dot{\underline{r}}|$

$\ddot{r} = |\ddot{\underline{r}}|$

(1) $$\ddot{\underline{r}} = -\frac{GM}{r^2}\frac{\underline{r}}{|\underline{r}|} \qquad / \cdot \frac{\underline{r}}{|\underline{r}|}$$

$$\ddot{r} = -\frac{GM}{r^2} \quad \Rightarrow \quad \ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{dr}\cdot\frac{dr}{dt} = \frac{d\dot{r}}{dr}\dot{r} = \frac{d}{dr}\left(\frac{1}{2}\dot{r}^2\right)$$

$$M = \frac{4\pi}{3} r(t)^3 \varrho(t) = \text{const.}$$

$$\frac{d}{dr}\left(\frac{1}{2}\dot{r}^2\right) = -\frac{GM}{r^2}$$

$$\frac{1}{2}\dot{r}^2 = \frac{GM}{r} + \text{const.} \quad \longrightarrow \quad \frac{1}{2}\dot{r}^2 - \frac{GM}{r} = \text{const.}$$

$$\dot{r}^2 = \frac{8\pi G\varrho}{3} r^2 + \text{const.}$$

At any given time t_0: $\dot{r}(t_0) = v_0$, $r(t_0) = r_0$ and $\text{const.} = \text{const.}\cdot r_0^2$

$$v_0^2 = \left(\frac{8\pi G\varrho(t_0)}{3} + \text{const.}\right) r_0^2 \Rightarrow \underline{v_0 = H(t_0)\, r_0}$$ Hubble-Lemaître law

$H(t)$ - Hubble parameter

$H_0 = H(t_0)$ - Hubble constant

$$\Rightarrow \quad \dot{r} = H(t)\, r$$

$$\frac{d}{dt}(\ln r) = H(t)$$

$$\ln r = \int_0^t H(t')dt' + \ln r_0$$

$$r(t) = r_0 \underbrace{e^{\int_0^t H(t')dt'}}_{= a(t) \text{ scale factor}} = a(t)\, r_0$$

$$\dot{r} = \dot{a}\, r_0 = \frac{\dot{a}}{a} \underbrace{a\, r_0}_{} = H(t)\, r$$

$$H(t) = \frac{\dot{r}}{r} = \frac{\dot{a}}{a}$$

Newtonian derivation of the Friedmann equations

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$$\Rightarrow \quad \frac{1}{2}\dot{r}^2 - \frac{GM}{r} = \text{const.} = \mathcal{E}$$

$$\frac{1}{2}\dot{r}^2 - \frac{4\pi G\rho}{3} r^2 = \mathcal{E}$$

$$\left(\frac{\dot{r}}{r}\right)^2 - \frac{8\pi G\rho}{3} = \frac{2\mathcal{E}}{r^2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2\mathcal{E}}{a^2 r_0^2}$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}}$$

FI

k - curvature parameter

$$k = -\frac{2\mathcal{E}}{c^2 r_0^2} = \text{const.}$$

$$\Rightarrow \quad \ddot{r} = -\frac{GM}{r^2} = -\frac{4\pi G\rho}{3} r$$

$$\ddot{a} r_0 = -\frac{4\pi G\rho}{3} a r_0$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho}$$

FII - if $p = \emptyset$!

$a(t) = ?$

$\rho(t) = ?$

Semi-relativistic approach: $M = \frac{E}{c^2} \Rightarrow \rho \leftrightarrow \rho + \frac{3p}{c^2}$

Equation of state (perfect fluid):

$p = w\rho c^2$ $\quad w$ - e.o.s parameter

$$\Rightarrow \quad \boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)} \rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1+3w)\rho$$

FII

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_{i=1}^{N}(1+3w_i)\rho_i$$

Newtonian derivation of the Friedmann equations

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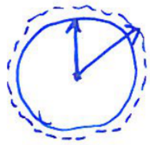
Conservation of energy:



$$dU = dQ + dW$$

Assuming adiabatic ($dQ = \emptyset$) expansion/contraction:

$$d\left(\rho c^2 \frac{4\pi r^3}{3}\right) = -p\, d\left(\frac{4\pi r^3}{3}\right)$$

$$d(\rho c^2 a^3 r_0^3) = -w \rho c^2 d(a^3 r_0^3)$$

$$a^3 d\rho + 3a^2 \rho\, da = -w\rho\, 3a^2 da$$

$$\frac{d\rho}{da} = -3(1+w)\frac{\rho}{a}$$

$$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$

$$d(\ln \rho) = d\left(\ln a^{-3(1+w)}\right)$$

$$\ln \rho = \ln a^{-3(1+w)} + \text{const.}$$

$$\boxed{\rho(a) = \rho_0\, a^{-3(1+w)}}$$

$w = \emptyset \Rightarrow$ matter $\quad \rho_m(a) = \rho_{m,0}\, a^{-3}$

$w = \frac{1}{3} \Rightarrow$ radiation $\quad \rho_r(a) = \rho_{r,0}\, a^{-4}$

$$\rho_m \sim a^{-3} \rightarrow \left(\rho_m = \frac{m}{V} = \frac{N\cdot m_0}{V} = \frac{N\cdot m_0}{\frac{4r^3\pi}{3}} = \frac{N\cdot m_0}{\frac{4\pi}{3}a^3 r_0^3} = \rho_{m,0}\, a^{-3}\right)$$

$$\rho_r \sim a^{-4}$$

$$\downarrow$$

$$\left(\rho_r = \frac{m}{V} = \frac{N\cdot \frac{hc}{\lambda}\cdot\frac{1}{c^2}}{V} = \frac{N\cdot \frac{hc}{a\lambda_0}\cdot\frac{1}{c^2}}{a^3 V_0} = \rho_{r,0}\, a^{-4}\right)$$

Special case: $w = -1 \rightarrow$ cosmological constant

$$\rho_\Lambda = \rho_{\Lambda,0}\, a^{\emptyset}$$

Matter: consists of non-relativistic particles ($E_0 \gg E_{kin}$)
↳ barionic matter (today), dark matter (cold)

Radiation: consists of relativistic particles ($E_0 \ll E_{kin}$)
↳ photons, GWs, neutrinos