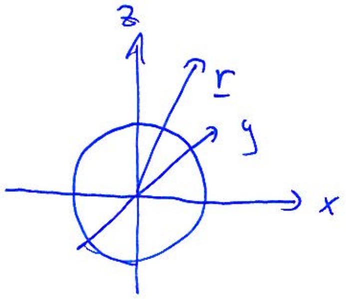

$\underline{v}(\underline{r},t)$ $\qquad \underline{e} = \frac{\underline{r}}{|\underline{r}|}$

$\rho(\underline{r},t)$ $\qquad r = |\underline{r}|$ $\qquad \underline{v}(\underline{r},t) = v(\underline{r},t)\underline{e}$

Euler equations (assuming $p=\emptyset$):

(1) $$\frac{\partial \rho}{\partial t} + \underline{\nabla}(\rho \underline{v}) = \emptyset$$

$$\frac{\partial \rho}{\partial t} + (\underline{\nabla}\rho)\underline{v} + \rho(\underline{\nabla}\,\underline{v}) = \emptyset$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial r} v + \rho \frac{1}{r^2}\frac{\partial (r^2 v)}{\partial r} = \emptyset$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial r} v + \rho \frac{1}{r^2}\left[2rv + r^2 \frac{\partial v}{\partial r}\right] = \emptyset$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial r} v + \frac{2\rho v}{r} + \rho \frac{\partial v}{\partial r} = \emptyset$$

(2) $$\frac{\partial \underline{v}}{\partial t} + (\underline{v}\,\underline{\nabla})\underline{v} = \underline{g} = -\frac{G}{r^2}\int_0^r 4\pi r'^2 \rho(r',t)\,dr' \cdot \underline{e} \qquad / \cdot \underline{e}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{G}{r^2}\int_0^r 4\pi r'^2 \rho\, dr'$$

initial condition: $\rho(\underline{r},\emptyset) = \rho_0 = \text{const.}$

$\underline{v}(\underline{r},\emptyset) = H_0 r \underline{e} \qquad (H_0 = \text{const.})$

Numerical solution:

$$v(\underline{r},t+\Delta t) = -\left[\frac{G}{r^2}\int_0^r 4\pi r'^2 \rho(r',t)\,dr' + v(\underline{r},t)\frac{\partial v(\underline{r},t)}{\partial r}\right]\Delta t + v(\underline{r},t)$$

$$\rho(\underline{r},t+\Delta t) = -\left[\frac{\partial \rho(\underline{r},t)}{\partial r} v(\underline{r},t) + \frac{2}{r}\rho(\underline{r},t)v(\underline{r},t) + \rho(\underline{r},t)\frac{\partial v(\underline{r},t)}{\partial r}\right]\Delta t + \rho(\underline{r},t)$$

$\boxed{t=\emptyset} \Rightarrow v(\underline{r},\Delta t) = -\left[\frac{4\pi G\rho_0 r}{3} + H_0^2 r\right]\Delta t + H_0 r = \left[H_0 - \left(\frac{4\pi G\rho_0}{3} + H_0^2\right)\Delta t\right] r$

$\rho(\underline{r},\Delta t) = -\left[2\rho_0 H_0 + \rho_0 H_0\right]\Delta t + \rho_0 = \rho_0(1 - 3H_0\Delta t)$

$\boxed{t=\Delta t} \Rightarrow v(r, 2\Delta t) = \ldots \quad \Rightarrow \boxed{\rho(\underline{r},t) = \rho(t)!!} \quad \boxed{\underline{v}(\underline{r},t) = H(t)\, r\, \underline{e}\ !!}$

$\Rightarrow$ (1) $\dot{\rho} + 2\rho H + \rho H = \emptyset$ $\qquad$ (2) $\dot{H}\cancel{r} + H^2 \cancel{r} = -\frac{4\pi G \rho \cancel{r}}{3}$

$\boxed{\dot{\rho} + 3\rho H = \emptyset}$ $\qquad$ $\boxed{3\dot{H} + 3H^2 = -4\pi G\rho}$

Let us choose a random comoving „dV" volume element at distance „r" within the sphere!
Its velocity satisfies:

$$\underline{v} = \underline{\dot{r}} = H\underline{r} \qquad (v = \dot{r} = Hr) \Rightarrow \dot{r} = Hr$$

$$d(\ln r) = H dt$$

$$\underbrace{r(t) = r_0 e^{\int_0 H(t)dt}}_{(\Rightarrow r(\phi) = r_0)}$$

Thus:

(1) $$\dot{\varrho} + 3\varrho\frac{\dot{r}}{r} = \phi$$

$$\frac{d}{dt}(\ln\varrho) = \frac{d}{dt}(\ln r^{-3})$$

$$\frac{d}{dt}(\ln \varrho r^3) = \phi$$

$$\varrho r^3 = \text{const.} \stackrel{!}{=} \frac{3M}{4\pi}$$

(2) $$3\dot{\left(\frac{\dot{r}}{r}\right)} + 3\frac{\dot{r}^2}{r^2} = -4\pi G\varrho$$

$$3\frac{\ddot{r}}{r} - \cancel{3\frac{\dot{r}^2}{r^2}} + \cancel{3\frac{\dot{r}^2}{r^2}} = -4\pi G\varrho$$

$$\boxed{\ddot{r} = -\frac{4\pi G\varrho r}{3} = -\frac{GM}{r^2}} \rightarrow \ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{dr}\cdot\frac{dr}{dt} = \frac{d\dot{r}}{dr}\dot{r} = \frac{d}{dr}\left(\frac{1}{2}\dot{r}^2\right)$$

$$\frac{d}{dr}\left(\frac{1}{2}\dot{r}^2\right) = -\frac{GM}{r^2}$$

$$\frac{1}{2}\dot{r}^2 = \frac{GM}{r} + \text{const.}$$

$$\boxed{\frac{1}{2}\dot{r}^2 - \frac{GM}{r} = \text{const.}}$$

$$\frac{r^2}{2}\left(\underbrace{\left(\frac{\dot{r}}{r}\right)^2}_{H^2} - \frac{8\pi G\varrho}{3}\right) = \text{const.} \stackrel{!}{=} \frac{r_0^2}{2}\left(H_0^2 - \frac{8\pi G\varrho_0}{3}\right) \quad \forall r_0$$

$$\text{const.} = \phi \Leftrightarrow H_0 = \sqrt{\frac{8\pi G\varrho_0}{3}} \Leftrightarrow \underbrace{\varrho_0 = \frac{3H_0^2}{8\pi G}}_{= \varrho_{crit}(\phi)}$$

$$\varrho_{crit}(t) = \frac{3H^2}{8\pi G}$$

$$\Omega(t) = \frac{\varrho(t)}{\varrho_{crit}(t)} \Rightarrow \frac{H^2 r^2}{2}(1 - \Omega) = \text{const.}$$

$$\text{const.} = \phi \Leftrightarrow \Omega = 1$$

Allowing $p \neq \emptyset$: $\quad p = p(\underline{r}, t)$

(2) $$\frac{\partial \underline{v}}{\partial t} + (\underline{v}\underline{\nabla})\underline{v} = \underline{g} - \underline{\nabla} p$$

In this case, we must also assume an equation of state: $\quad p \stackrel{!}{=} f(\rho)$

Equation of state of a perfect fluid:

$p = w \rho c^2$ where $w = \emptyset$ for non-relativistic matter

$w = \frac{1}{3}$ for relativistic matter

(i.e. $w \in [0; \frac{1}{3}]$ for any "non-exotic" matter types)

(Perfect fluid: no shear stresses, viscosity, heat conduction.)

Thus.: $\quad \underline{\nabla} p = w c^2 \underline{\nabla} \rho$

$\Rightarrow \underline{\nabla} p = \frac{\partial p}{\partial r} = w c^2 \frac{\partial \rho}{\partial r}$ $\Rightarrow$ This term would not change the solution!