$\boxed{\text{Derivation of the FLRW-metric}}$

Peter Raffai
peter.raffai@ttk.elte.hu

$$ds^2 = -d(ct)^2 + a(t)^2 R_0^2 \Big[\underbrace{d\chi^2 + S^2(\chi)\, d\theta^2 + S^2(\chi) \sin^2\theta \, d\phi^2}_{d\sigma^2} \Big]$$

$$d\sigma^2 \Longrightarrow g_{ij} \qquad x = (x^1, x^2, x^3) = (\chi, \theta, \phi)$$

$$g_{ij} = \begin{pmatrix} 1 & & \\ & S^2 & \\ & & S^2 \sin^2\theta \end{pmatrix} \qquad g_{ij} g^{ij} \overset{!}{=} 3 \Rightarrow g^{ij} = \begin{pmatrix} 1 & & \\ & \frac{1}{S^2} & \\ & & \frac{1}{S^2 \sin^2\theta} \end{pmatrix}$$

$$R = R_{ij} g^{ij}$$ [Ricci scalar of 3D space]

$$R_{ij} = \Gamma^k_{\ ij,k} - \Gamma^k_{\ ik,j} + \Gamma^k_{\ lk} \Gamma^l_{\ ij} - \Gamma^k_{\ lj} \Gamma^l_{\ ik} \qquad \Gamma^k_{\ ij,k} = \frac{\partial \Gamma^k_{\ ij}}{\partial x^k}$$

$$\Gamma^i_{\ kl} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{kj}}{\partial x^l} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^j} \right)$$

$$\Gamma^1_{\ 22} = -\frac{1}{2} \cdot 1 \cdot 2SS' = -SS' \qquad S' = \frac{dS}{d\chi}$$

$$\Gamma^1_{\ 33} = -\frac{1}{2} \cdot 1 \cdot 2SS' \sin^2\theta = -SS' \sin^2\theta$$

$$\Gamma^2_{\ 12} = \Gamma^2_{\ 21} = \frac{1}{2} S^{-2} \cdot 2SS' = \frac{S'}{S}$$

$$\Gamma^2_{\ 33} = -\frac{1}{2} \cdot \frac{1}{S^2} \cdot S^2 2 \sin\theta \cos\theta = -\sin\theta \cos\theta$$

$$\Gamma^3_{\ 31} = \Gamma^3_{\ 13} = \frac{1}{2} \frac{1}{S^2 \sin^2\theta} \cdot 2SS' \sin^2\theta = \frac{S'}{S}$$

$$\Gamma^3_{\ 23} = \Gamma^3_{\ 32} = \frac{1}{2} \frac{1}{S^2 \sin^2\theta} \cdot S^2 2 \sin\theta \cos\theta = \frac{\cos\theta}{\sin\theta}$$

$$R_{11} = -\Gamma^2_{\ 12,1} - \Gamma^3_{\ 13,1} - (\Gamma^2_{\ 21})^2 - (\Gamma^3_{\ 31})^2 = -\frac{S''S - S'S'}{S^2} 2 - 2\frac{S'^2}{S^2} = -\frac{2S''}{S}$$

$$R_{22} = \Gamma^1_{\ 22,1} - \Gamma^3_{\ 23,2} - (\Gamma^3_{\ 32})^2 = -S'^2 - SS'' - \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta} = -S'^2 - SS'' + 1$$

$$R_{33} = \Gamma^1_{\ 33,1} + \Gamma^2_{\ 33,2} - \Gamma^2_{\ 33} \Gamma^3_{\ 32} = -S'^2 \sin^2\theta - SS'' \sin^2\theta - \cancel{\cos^2\theta} + \sin^2\theta + \cancel{\cos^2\theta} = (-S'^2 - SS'' + 1) \sin^2\theta$$

$$R = R_{11} g^{11} + R_{22} g^{22} + R_{33} g^{33} = -\frac{2S''}{S} - \frac{2S'^2}{S^2} - \frac{2S''}{S} + \frac{2}{S^2} = -\frac{4S''}{S} - \frac{2S'^2}{S^2} + \frac{2}{S^2}$$

Peter Raffai
peter.raffai@ttk.elte.hu

$$\frac{dR}{dx} \stackrel{!}{=} \emptyset$$

$$-4\,\frac{s'''s - s's''}{s^2} - 2\,\frac{2s's''s^2 - 2ss'^3}{s^4} - \frac{4s'}{s^3} \stackrel{!}{=} \emptyset$$

$$\frac{s'''}{s} - \frac{s's''}{s^2} + \frac{s's''}{s^2} - \frac{s'^3}{s^3} + \frac{s'}{s^3} = \emptyset$$

$$s^2 s''' - s'^3 + s' = \emptyset \quad \longrightarrow \quad s' = \emptyset \Rightarrow \left.\begin{matrix} s = \text{const.} \\ s(\emptyset) \stackrel{!}{=} \emptyset \end{matrix}\right\} s = \emptyset \ \lightning$$

$$\Downarrow$$

$$s' = f \neq \emptyset$$

$$\longrightarrow f(s(x)) \stackrel{!}{=} s'(x) \qquad \Big/ \frac{d}{dx}$$

$$\longrightarrow \dot{f}s' = s'' \qquad \Big/ \frac{d}{dx} \qquad \left(\dot{f} = \frac{df}{ds}\right)$$

$$\longrightarrow \ddot{f}s'^2 + \dot{f}s'' = s'''$$

$$\hookrightarrow s^2(\ddot{f}s'^2 + \dot{f}s'') - f^3 + f = \emptyset$$

$$s^2(\ddot{f}f^2 + \dot{f}^2 f) - f^3 + f = \emptyset$$

$$\ddot{f}f^2 + \dot{f}^2 f + \frac{f - f^3}{s^2} = \emptyset$$

$$\underbrace{\ddot{f}f + \dot{f}^2}_{\left(\frac{1}{2}f^2\right)^{\cdot\cdot}} + \frac{1 - f^2}{s^2} = \emptyset$$

$$\left(\tfrac{1}{2}f^2\right)^{\cdot\cdot} + \frac{1 - f^2}{s^2} = \emptyset$$

$$\longrightarrow f^2 \stackrel{!}{=} g \quad \Longrightarrow \quad \ddot{g} = \frac{2}{s^2}(g - 1) \qquad (g = g(s))$$

$$\longrightarrow h = g - 1 \quad \Longrightarrow \quad \underbrace{\ddot{h} = \frac{2}{s^2}h} \qquad (h = h(s))$$

Let h be $h(s) = s^\alpha$ $\quad \alpha(\alpha - 1) = 2 \Rightarrow \alpha^2 - \alpha - 2 = \emptyset$

$$(\alpha - 2)(\alpha + 1) = \emptyset \Rightarrow \begin{matrix} \alpha_1 = -1 \\ \alpha_2 = 2 \end{matrix}$$

$$\Longrightarrow h(s) = \frac{c_1}{s} + c_2 s^2$$

$$g(s) = 1 + \frac{c_1}{s} + c_2 s^2 = f^2$$

$$f(s) = \pm\sqrt{1 + \frac{c_1}{s} + c_2 s^2} = s' \longrightarrow s(\emptyset) \stackrel{!}{=} \emptyset \Rightarrow f(\emptyset) = s'(\emptyset) = \begin{cases} 1 & \text{if } c_1 = \emptyset \\ \pm\infty & \text{if } c_1 \neq \emptyset \end{cases}$$

$$\Downarrow$$

$$c_1 \stackrel{!}{=} \emptyset$$

$$\boxed{s' = \pm\sqrt{1 + c_2 s^2}}$$

Peter Raffai
peter.raffai@ttk.elte.hu

$$s' = \pm\sqrt{1 + c_2 s^2}$$

(1) $c_2 = \omega^2 > 0$

(2) $c_2 = 0$

(3) $c_2 = -\Omega^2 < 0$

(1) $c_2 = \omega^2 > 0$

$$s' = \sqrt{1+\omega^2 s^2} \longrightarrow \chi = \int \frac{ds}{\sqrt{1+\omega^2 s^2}} = \frac{1}{\omega}\,\mathrm{arsinh}(\omega s) + \underbrace{\mathrm{const.}}_{= 0}$$

since $\chi(0) \overset{!}{=} 0$

$$s(\chi) = \frac{1}{\omega}\,\mathrm{sh}(\omega\chi)$$

$$\Longrightarrow \boxed{s(\chi) = \mathrm{sh}(\chi)}$$

(2) $c_2 = 0$

$$s' = \pm 1 \Longrightarrow s(\chi) = \underbrace{\mathrm{const.}}_{= 0} \pm \chi$$

since $s(0) \overset{!}{=} 0$

$$\Longrightarrow \boxed{s(\chi) = \chi}$$

(3) $c_2 = -\Omega^2 < 0$

$$s' = \sqrt{1-\Omega^2 s^2} \longrightarrow \chi = \int \frac{ds}{\sqrt{1-\Omega^2 s^2}} = \frac{1}{\Omega}\arcsin(\Omega s) + 0$$

$$s(\chi) = \frac{1}{\Omega}\sin(\Omega\chi)$$

$$\Longrightarrow \boxed{s(\chi) = \sin(\chi)}$$