# Linear evolution of matter density fluctuations



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→ Newtonian approximation applied

$$\rho(\underline{r},t) = \bar{\rho}(t) + \delta\rho(\underline{r},t)$$

$$\underline{v}(\underline{r},t) = \bar{v}(\underline{r},t) + \delta v(\underline{r},t) \quad \rightarrow \quad \bar{v}(\underline{r},t) = H(t)\underline{r}$$

$$p(\underline{r},t) = \bar{p}(t) + \delta p(\underline{r},t)$$

$$\phi(\underline{r},t) = \bar{\phi}(\underline{r},t) + \delta\phi(\underline{r},t)$$

(1) Continuity equation (i.e. energy conservation)

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}(\rho\underline{v}) = 0 \quad \Rightarrow \quad \frac{\partial\bar{\rho}}{\partial t} + \frac{\partial\delta\rho}{\partial t} + \underline{\nabla}\left[(\bar{\rho}+\delta\rho)(\bar{v}+\delta v)\right] = 0$$

$$\cancel{\frac{\partial\bar{\rho}}{\partial t}} + \frac{\partial\delta\rho}{\partial t} + \underline{\nabla}(\cancel{\bar{\rho}\bar{v}} + \delta\rho\bar{v} + \bar{\rho}\delta v + \underbrace{\delta\rho\delta v}_{\approx 0}) = 0$$

$$\frac{\partial\delta\rho}{\partial t} + \underline{\nabla}(\delta\rho\bar{v}) + \bar{\rho}\underline{\nabla}\delta v = 0$$

$$\frac{\partial\delta\rho}{\partial t} + \bar{v}\underline{\nabla}\delta\rho + 3H\delta\rho + \bar{\rho}\underline{\nabla}\delta v = 0$$

↳ $\underline{x} = \dfrac{\underline{r}}{a} \Rightarrow \underline{r} = a\underline{x}$

$\underline{u} = \dfrac{\delta v}{a}$

↳ $$\delta = \frac{\delta\rho}{\bar{\rho}} \Rightarrow \frac{\partial\delta}{\partial t} = \frac{\frac{\partial\delta\rho}{\partial t}\bar{\rho} - \frac{\partial\bar{\rho}}{\partial t}\delta\rho}{\bar{\rho}^2} = \frac{\partial\delta\rho}{\partial t}\frac{1}{\bar{\rho}} - \frac{\partial\bar{\rho}}{\partial t}\delta\frac{1}{\bar{\rho}} =$$

$$= \frac{\partial\delta\rho}{\partial t}\frac{1}{\bar{\rho}} + \underline{\nabla}(\bar{\rho}\bar{v})\frac{\delta}{\bar{\rho}} = \frac{\partial\delta\rho}{\partial t}\frac{1}{\bar{\rho}} + 3H\delta$$

$$\Rightarrow \frac{\partial\delta\rho}{\partial t}\frac{1}{\bar{\rho}} = \frac{\partial\delta}{\partial t} - 3H\delta$$

↳ $$\left.\frac{\partial\delta\rho}{\partial t}\right|_x = \left.\frac{\partial\delta\rho}{\partial t}\right|_r + \left.\frac{\partial\underline{r}}{\partial t}\right|_x \underline{\nabla}_r\delta\rho = \left.\frac{\partial\delta\rho}{\partial t}\right|_r + \dot{a}\underline{x}\frac{1}{a}\underline{\nabla}_x\delta\rho = \left.\frac{\partial\delta\rho}{\partial t}\right|_r + Hx\frac{\partial\delta\rho}{\partial x} \Rightarrow$$

$$\Rightarrow \left.\frac{\partial\delta\rho}{\partial t}\right|_r = \left.\frac{\partial\delta\rho}{\partial t}\right|_x - Hx\frac{\partial\delta\rho}{\partial x}$$

$$\Rightarrow \frac{\partial\delta\rho}{\partial t}\frac{1}{\bar{\rho}} + \bar{v}\underline{\nabla}_r\delta + 3H\delta + \underline{\nabla}_r\delta v = 0$$

$$\left(\frac{\partial\delta\rho}{\partial t} - Hx\frac{\partial\delta\rho}{\partial x}\right)\frac{1}{\bar{\rho}} + \bar{v}\underline{\nabla}_r\delta + 3H\delta + \underline{\nabla}_r\delta v = 0$$

$$\frac{\partial\delta}{\partial t} \cancel{- 3H\delta} \cancel{- Hx\frac{\partial\delta}{\partial x}} \cancel{+ Hx\frac{\partial\delta}{\partial x}} \cancel{+ 3H\delta} + \underline{\nabla}_x\underline{u} = 0$$

$$\boxed{\frac{\partial\delta}{\partial t} + \underline{\nabla}_x\underline{u} = 0}$$

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(2) Euler's equation

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v}\underline{\nabla})\underline{v} = -\underline{\nabla}\phi - \frac{\underline{\nabla} p}{\varrho}$$

$$\Rightarrow \frac{\partial}{\partial t}(\bar{v}+\delta v) + [(\bar{v}+\delta v)\underline{\nabla}](\bar{v}+\delta v) = -\underline{\nabla}(\bar{\phi}+\delta\phi) - \frac{\underline{\nabla}(\bar{p}+\delta p)}{\bar{\varrho}}$$

↳ Here we ignore $\delta\varrho$.

$$\frac{\partial \bar{v}}{\partial t} + \frac{\partial \delta v}{\partial t} + (\bar{v}\underline{\nabla})\bar{v} + (\bar{v}\underline{\nabla})\delta v + (\delta v\underline{\nabla})\bar{v} = -\underline{\nabla}\bar{\phi} - \underline{\nabla}\delta\phi - \frac{\underline{\nabla}\bar{p}}{\bar{\varrho}} - \frac{\underline{\nabla}\delta p}{\bar{\varrho}}$$

$$\frac{\partial \delta v}{\partial t} + Hr\frac{\partial \delta v}{\partial r} + H\delta v = -\nabla\delta\phi - \frac{\nabla \delta p}{\bar{\varrho}}$$

$$\underline{x} = \frac{r}{a}$$

$$\frac{\partial \delta v}{\partial t} - Hx\frac{\partial \delta v}{\partial x} + Hx\frac{\partial \delta v}{\partial x} + H\delta v = -\frac{\underline{\nabla}_x\delta\phi}{a} - \frac{\underline{\nabla}_x \delta p}{a\bar{\varrho}}$$

$$\underline{u} = \frac{\delta v}{a} \Rightarrow \delta v = a\underline{u} \qquad \frac{\partial \delta v}{\partial t} = \dot{a}\underline{u} + a\frac{\partial \underline{u}}{\partial t} = Ha\underline{u} + a\frac{\partial \underline{u}}{\partial t}$$

$$\Rightarrow Ha\underline{u} + a\frac{\partial \underline{u}}{\partial t} + Ha\underline{u} = -\frac{\underline{\nabla}_x\delta\phi}{a} - \frac{\underline{\nabla}_x\delta p}{a\bar{\varrho}}$$

$$\boxed{\frac{\partial \underline{u}}{\partial t} + 2H\underline{u} = -\frac{\underline{\nabla}_x\delta\phi}{a^2} - \frac{\underline{\nabla}_x\delta p}{a^2\bar{\varrho}}}$$

(3) Poisson's equation

$$\Delta\phi = 4\pi G\varrho \Rightarrow \Delta(\cancel{\bar{\phi}} + \delta\phi) = 4\pi G(\cancel{\bar{\varrho}} + \delta\varrho)$$

$$\Delta\delta\phi = 4\pi G\,\delta\varrho$$

$$\boxed{\Delta_x\delta\phi = 4\pi G\bar{\varrho}a^2\delta}$$

$$\Rightarrow \frac{\partial}{\partial t}(1) \Rightarrow \frac{\partial^2\delta}{\partial t^2} + \underline{\nabla}_x\frac{\partial \underline{u}}{\partial t} = \emptyset \Rightarrow \underline{\nabla}_x\frac{\partial \underline{u}}{\partial t} = -\frac{\partial^2\delta}{\partial t^2}$$

$$\Rightarrow \underline{\nabla}_x(2) \Rightarrow \underline{\nabla}_x\frac{\partial \underline{u}}{\partial t} + 2H\underline{\nabla}_x\underline{u} = -\frac{\Delta_x\delta\phi}{a^2} - \frac{\Delta_x\delta p}{a^2\bar{\varrho}}$$

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$$\frac{\partial^2\delta}{\partial t^2} + 2H\frac{\partial\delta}{\partial t} = 4\pi G\bar{\varrho}\delta + \frac{\Delta_x\delta p}{a^2\bar{\varrho}}$$

↳ $\frac{\delta p}{\delta\varrho} = c_s^2 (\approx wc^2)$

$$\frac{\partial^2\delta}{\partial t^2} + 2H\frac{\partial\delta}{\partial t} = 4\pi G\bar{\varrho}\delta + \frac{c_s^2}{a^2}\Delta_x\delta$$

$$\delta(\underline{r},t) \stackrel{!}{=} \delta_0 e^{i(\omega t - \underline{k}'\underline{r})} = \delta_0 e^{i(\omega t - \underline{k}\underline{x})}$$

$$\Rightarrow \Delta_x\delta = -k^2\delta \qquad \rightarrow k = |\underline{k}| = \frac{2\pi a}{\lambda}$$ comoving wave number

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$$\frac{\partial^2\delta}{\partial t^2} + 2H\frac{\partial\delta}{\partial t} = \left(4\pi G\bar{\rho} - c_s^2 k^2 a^{-2}\right)\delta$$

$$4\pi G\bar{\rho} = c_s^2 \frac{4\pi^2}{\lambda_J^2}$$

$$\cancel{4\pi G}\frac{\bar{\rho}}{\rho_c}\cdot\frac{3H^2}{\cancel{8\pi G}_{2}} = c_s^2\frac{4\pi^2}{\lambda_J^2} \qquad \rightarrow \quad \frac{\bar{\rho}}{\rho_c} = \Omega_m$$

$$\lambda_J^2 = \frac{8\pi^2 c_s^2}{3\Omega_m H^2}$$

$$\lambda_J = 2\pi\sqrt{\frac{2}{3\Omega_m}}\left(\frac{c_s}{c}\right)\cdot\left(\frac{c}{H}\right) \quad \rightarrow \text{ Jeans' wavelength}$$

$\frac{c}{H}$: Hubble radius

$$\boxed{\frac{\partial^2\delta}{\partial t^2} + 2H\frac{\partial\delta}{\partial t} = \frac{3}{2}\Omega_m H^2\left[1 - \left(\frac{\lambda_J}{\lambda}\right)^2\right]\delta}$$

$\lambda_J << \lambda$

↳ Radiation-dominated era, $\Omega_m \simeq 0$ $(k=0)$

$$\left.\begin{aligned} a &\sim t^{\frac{1}{2}} \\ \dot{a} &\sim \frac{1}{2}t^{-\frac{1}{2}} \end{aligned}\right\} \frac{\dot{a}}{a} = \frac{1}{2}\frac{1}{t} = H$$

$$\frac{\partial^2\delta}{\partial t^2} + \frac{1}{t}\frac{\partial\delta}{\partial t} = 0 \quad \Rightarrow \quad \delta(t) = \delta_0 \ln\left(\frac{t}{t_0}\right)$$

$$\delta(a) = 2\delta_0 \ln\left(\frac{a}{a_0}\right) \quad \Rightarrow \delta(a) \sim \ln a$$

↳ Matter-domonated era, $\Omega_m \simeq 1$ $(k=0)$

$$\left.\begin{aligned} a &\sim t^{\frac{2}{3}} \\ \dot{a} &\sim \frac{2}{3}t^{-\frac{1}{3}} \end{aligned}\right\} \frac{\dot{a}}{a} = \frac{2}{3}\frac{1}{t} = H$$

$$\frac{\partial^2\delta}{\partial t^2} + \frac{4}{3}\frac{1}{t}\frac{\partial\delta}{\partial t} - \frac{3}{2}\cdot 1\cdot\frac{4}{9}\cdot\frac{1}{t^2}\delta = 0$$

$$\frac{\partial^2\delta}{\partial t^2} + \frac{4}{3}\frac{1}{t}\frac{\partial\delta}{\partial t} - \frac{2}{3}\frac{1}{t^2}\delta = 0$$

↳ $\delta(a) = \delta_0' a^n = \delta_0 t^{\frac{2}{3}n}$ ↳ $\frac{\partial\delta}{\partial t} = \delta_0\frac{2}{3}n t^{\frac{2}{3}n-1}$

↳ $\frac{\partial^2\delta}{\partial t^2} = \delta_0\frac{2}{3}n\left(\frac{2}{3}n-1\right)t^{\frac{2}{3}n-2}$

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$$\Rightarrow \frac{2}{3}n\left(\frac{2}{3}n-1\right)+\frac{8}{9}n-\frac{2}{3}=0$$

$$\frac{4}{9}n^2-\frac{2}{3}n+\frac{8}{9}n-\frac{2}{3}=0$$

$$n^2+\frac{1}{2}n-\frac{3}{2}=0 \longrightarrow n=\frac{-\frac{1}{2}\pm\sqrt{\frac{1}{4}+4\cdot\frac{3}{2}}}{2} \begin{cases} n_+=1 \\ n_-=-\frac{3}{2}\end{cases} \quad \begin{aligned} \delta(a)&=D_+(a)\delta_+ + D_-(a)\delta_- \\ \delta(a)&=\delta_+ a+\delta_- a^{-\frac{3}{2}} \\ \delta(t)&=\delta_+ t^{\frac{2}{3}}+\delta_- t^{-1}\end{aligned}$$

$D_+(a)=a$ (linear growth factor)

↳ $\Lambda$-dominated era, $\Omega_m \simeq 0$ $(k=0)$

$$\left.\begin{aligned} a &\sim e^{Ht} \\ \dot{a} &\sim He^{Ht}\end{aligned}\right\} \frac{\dot{a}}{a}=H=\text{const.}$$

$$\frac{\partial^2\delta}{\partial t^2}+2H\frac{\partial\delta}{\partial t}=0 \quad \Rightarrow \quad \delta(t)=\delta_+'+\delta_-' e^{-2Ht}$$

$$\delta(a)=\delta_+ + \delta_- a^{-2}$$

$$D_+(a)=a^0=1 \quad (\delta_+=\text{const.})$$

$\boxed{\lambda_J \gg \lambda}$

↳ Matter-dominated era, $\Omega_m \simeq 1$ $(k=0)$

In plasma: $c_s^2 \simeq \frac{1}{3}c^2$ $\left(c_s=\frac{c}{\sqrt{3\left(1+\frac{3\Omega_b}{4\Omega_r}\right)}}\right)$

$$\lambda_J=2\pi\sqrt{\frac{2}{3}}\frac{1}{\sqrt{3}}\cdot\frac{c}{H}\simeq 3\frac{c}{H}$$

$$\frac{\partial^2\delta}{\partial t^2}+\frac{4}{3}\frac{1}{t}\frac{\partial\delta}{\partial t}=-\frac{1}{3}c^2k'^2\delta$$

$$\omega^2=\frac{1}{3}c^2k'^2$$

$$\frac{\cancel{4\pi^2}}{T^2}=\frac{1}{3}c^2\frac{\cancel{4\pi^2}}{\lambda^2} \quad \Rightarrow \quad \boxed{T=\frac{\sqrt{3}}{c}\lambda}$$

# $\boxed{\text{Mészáros - effect}}$

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Based on:
Mészáros, P.; A&A 37., 2, p. 225-228. (1974)

$\boxed{\lambda_J \ll \lambda}$

(1) $\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G \rho_m \delta = \emptyset$

(2) $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_r)$

(3) $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_m + 2\rho_r)$

(4) $\rho_{m,0}\, a_{eq}^{-3} = \rho_{r,0}\, a_{eq}^{-4}$

$$\frac{\rho_{m,0}}{\rho_{r,0}} = \frac{1}{a_{eq}}$$

$$y = \frac{\rho_m}{\rho_r} = \frac{\rho_{m,0}\, a^{-3}}{\rho_{r,0}\, a^{-4}} = \frac{a}{a_{eq}}$$

$$\rightarrow \dot{\delta} = \frac{\partial \delta}{\partial a}\dot{a}$$

$$\rightarrow \ddot{\delta} = \left(\frac{\partial \delta}{\partial a}\dot{a}\right)^{\bullet} = \left(\frac{\partial \delta}{\partial a}\right)^{\bullet}\dot{a} + \frac{\partial \delta}{\partial a}\ddot{a} = \frac{\partial^2 \delta}{\partial a^2}\dot{a}^2 + a\frac{\partial \delta}{\partial a}\frac{\ddot{a}}{a}$$

$$\Rightarrow a^2 \frac{\partial^2 \delta}{\partial a^2}\left(\frac{\dot{a}}{a}\right)^2 + a\frac{\partial \delta}{\partial a}\frac{\ddot{a}}{a} + 2a\left(\frac{\dot{a}}{a}\right)^2\frac{\partial \delta}{\partial a} - 4\pi G \rho_m \delta = \emptyset$$

$$y^2\delta'' + y\delta' \cdot \frac{-\cancel{\frac{4\pi G}{3}}(\rho_m + 2\rho_r)}{\cancel{\frac{8\pi G}{3}}(\rho_m + \rho_r)} + 2y\delta' - \frac{\cancel{4\pi G}\rho_m}{\frac{\cancel{8\pi G}}{3}(\rho_m + \rho_r)}\delta = \emptyset \qquad \left[\delta' = \frac{\partial \delta}{\partial y}\right]$$

$$y^2\delta'' - y\delta'\frac{1}{2}\frac{y+2}{y+1} + 2y\delta' - \frac{3}{2}\frac{y}{y+1}\delta = \emptyset$$

$$\delta'' - \frac{y+2}{2y(y+1)}\delta' + \frac{2}{y}\delta' - \frac{3}{2y(y+1)}\delta = \emptyset$$

$$\delta'' - \frac{y+2}{2y(y+1)}\delta' + \frac{4(y+1)}{2y(y+1)}\delta' - \frac{3}{2y(y+1)}\delta = \emptyset$$

$$\delta'' + \frac{3y+2}{2y(y+1)}\delta' - \frac{3}{2y(y+1)}\delta = \emptyset \quad \text{(Mészáros equation)}$$

Solution: $\delta(y) = \delta_0\left(1 + \frac{3}{2}y\right)$

Radiation dominated epoch ($y \gtrsim \emptyset$):

$$\delta(y) = \delta_0 = \text{const.} \qquad (\delta \sim a^0)$$

Matter dominated epoch ($y \gg 1$):

$$\delta(y) \approx \delta_0 y \quad \rightarrow \quad \delta(a) = \delta_0 a \quad (\delta \sim a)$$

# Growth of matter density fluctuations



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Flat FLRW-spacetime with scalar perturbations:

$$ds^2 = -(1+2\Psi(\underline{x},t))d(ct)^2 + a^2(t)(1+2\phi(\underline{x},t))d\underline{x}^2$$

$\underline{x}$: comoving position vector

$\Psi$: gravitational potential

Conformal time: $d\eta = \frac{dt}{a(t)} = \frac{da}{a^2 H} = \frac{d(\ln a)}{aH}$

Primes denote derivative w.r.t. conformal time.

Particle velocity and momentum:

$$\underline{v} = \frac{\underline{p}}{m} = a\frac{d\underline{x}}{dt} = \underline{x}'$$

Fluid velocity and divergence: $\underline{u}$ ; $\theta = \partial_i u^i$ $\quad (\partial_i = \frac{\partial}{\partial x^i})$

No interactions for dark matter and we neglect interactions for baryons.

Collisionless Boltzmann equation describes the evolution of probability density function $f_m(t,\underline{x},\underline{p})$ in phase space:

$$\frac{df_m}{dt} = \frac{\partial f_m}{\partial t} + \frac{\partial f_m}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f_m}{\partial p^i}\frac{dp^i}{dt} = 0$$

where $\frac{dx^i}{dt} = \frac{p^i}{am}$ $\qquad \frac{dp^i}{dt} = -Hp^i - \frac{m}{a}\partial_i\Psi$

and $\delta_m(t,\underline{x}) = \frac{\rho_m(t,\underline{x}) - \bar{\rho}_m(t)}{\bar{\rho}_m(t)} = \frac{\delta\rho_m(t,\underline{x})}{\bar{\rho}_m(t)}$

$$\Delta\Psi = 4\pi G a^2 \delta\rho_m = 4\pi G a^2 \bar{\rho}_m \delta_m = \frac{1}{2}\frac{8\pi G}{3H^2}\bar{\rho}_m 3H^2 a^2 \delta_m = \frac{3}{2}\Omega_m a^2 H^2 \delta_m$$

$$\Rightarrow \frac{df_m}{dt} = \frac{\partial f_m}{\partial t} + \frac{\partial f_m}{\partial x^i}\frac{p^i}{am} - \frac{\partial f_m}{\partial p^i}\left(Hp^i + \frac{m}{a}\frac{\partial\Psi}{\partial x^i}\right) = 0$$

For <u>cold</u> matter:

$$f_m(t,\underline{x},\underline{p}) = \frac{\rho_m(t,\underline{x})}{m}(2\pi)^3\delta_D^{(3)}(\underline{p} - m\underline{u}_m(t,\underline{x})) \quad \text{(i.e. no velocity dispersion)}$$

We only solve the Boltzmann equation by taking its moments:

$$\langle A\rangle_{f_m}(t,\underline{x}) = \int\frac{d^3\underline{p}}{(2\pi)^3}A(t,\underline{x},\underline{p})\,f_m(t,\underline{x},\underline{p})$$

Zeroth moment:

$$\langle 1\rangle_{f_m}(t,\underline{x}) = \int\frac{d^3p}{(2\pi)^3} f_m(t,\underline{x},\underline{p}) = \int\frac{d^3p}{(2\pi)^3}\frac{\rho_m(t,\underline{x})}{m}(2\pi)^3\delta_D^{(3)}(\underline{p}-m\underline{u}_m(t,\underline{x})) =$$

$$= \frac{\rho_m(t,\underline{x})}{m} = n(t,\underline{x}) \qquad \text{(particle number density)}$$

First moment:

$$\langle p^i\rangle_{f_m}(t,\underline{x}) = \int\frac{d^3p}{(2\pi)^3} p^i f_m(t,\underline{x},\underline{p}) = \int\frac{d^3p}{(2\pi)^3}\frac{\rho_m(t,\underline{x})}{m}(2\pi)^3 p^i\delta_D^{(3)}(\underline{p}-m\underline{u}_m(t,\underline{x})) =$$

$$= \rho_m(t,\underline{x})\,u^i_m(t,\underline{x})$$

Second moment:

$$\langle p^ip^j\rangle_{f_m}(t,\underline{x}) = \int\frac{d^3p}{(2\pi)^3} p^ip^j f_m(t,\underline{x},\underline{p}) = \int\frac{d^3p}{(2\pi)^3}\frac{\rho_m(t,\underline{x})}{m}(2\pi)^3 p^ip^j\delta_D^{(3)}(\underline{p}-m\underline{u}_m(t,\underline{x})) =$$

$$= m\rho_m(t,\underline{x})\,u^i_m(t,\underline{x})\,u^j_m(t,\underline{x})$$

[In general: $\langle p^ip^j\rangle_{f_m} = m\rho_m u^i_m u^j_m + m\sigma^{ij}_m$]

anisotropic stress from velocity dispersion

$$\overset{\int}{\Longrightarrow}\ \frac{\partial}{\partial t}\langle 1\rangle_{f_m} + \frac{1}{am}\frac{\partial}{\partial x^i}\langle p^i\rangle_{f_m} - H\Big(\underbrace{\langle 0\rangle_{f_m}}_{=0} - \underbrace{\Big\langle\frac{\partial p^i}{\partial p^i}\Big\rangle_{f_m}}_{=\langle 3\rangle_{f_m}}\Big) - \frac{m}{a}\frac{\partial\Psi}{\partial x^i}\underbrace{\frac{\partial}{\partial p^i}\langle 1\rangle_{f_m}}_{=0} = 0$$

$$\frac{1}{\cancel{m}}\frac{\partial\rho_m}{\partial t} + \frac{1}{a\cancel{m}}\frac{\partial\rho_m}{\partial x^i}u^i_m + \frac{1}{a\cancel{m}}\rho_m\frac{\partial u^i_m}{\partial x^i} + 3H\frac{\rho_m}{\cancel{m}} = 0$$

$$\delta_m(t,\underline{x}) = \frac{\rho_m(t,\underline{x}) - \bar\rho_m(t)}{\bar\rho_m(t)} \Rightarrow \rho_m(t,\underline{x}) = \bar\rho_m(t)(1+\delta_m(t,\underline{x})) = \bar\rho_m(t) + \bar\rho_m(t)\delta_m(t,\underline{x})$$

$$\cancel{\frac{\partial\bar\rho_m}{\partial t}} + \cancel{\delta_m\frac{\partial\bar\rho_m}{\partial t}} + \cancel{\bar\rho_m}\frac{\partial\delta_m}{\partial t} + \frac{1}{a}u^i_m\cancel{\bar\rho_m}\frac{\partial\delta_m}{\partial x^i} + \frac{1}{a}\cancel{\bar\rho_m}\frac{\partial u^i_m}{\partial x^i} + \frac{1}{a}\cancel{\bar\rho_m}\delta_m\frac{\partial u^i_m}{\partial x^i} + \cancel{3H\bar\rho_m} + \cancel{3H\bar\rho_m\delta_m} = 0$$

$$\delta_m' + \frac{\partial}{\partial x^i}(u^i_m\delta_m) + \frac{\partial}{\partial x^i}(u^i_m) = 0$$

$$\boxed{\delta_m' + \frac{\partial}{\partial x^i}\big((1+\delta_m)u^i_m\big) = 0} \qquad (1)$$

$$\overset{\int p\cdot}{\Longrightarrow}\ \frac{\partial}{\partial t}\langle p^j\rangle_{f_m} + \frac{1}{am}\frac{\partial}{\partial x^i}\langle p^ip^j\rangle_{f_m} + H\Big\langle\underbrace{\frac{\partial}{\partial p^i}(p^ip^j)}_{3p^j+p^j=4p^j}\Big\rangle_{f_m} + \frac{m}{a}\frac{\partial\Psi}{\partial x^i}\Big\langle\underbrace{\frac{\partial}{\partial p^i}(p^j)}_{\delta_{ij}}\Big\rangle_{f_m} = 0$$

$$\frac{\partial}{\partial t}(\rho_m u^j_m) + \frac{1}{am}\frac{\partial}{\partial x^i}\langle p^ip^j\rangle_{f_m} + 4H\langle p^j\rangle_{f_m} + \frac{m}{a}\frac{\partial\Psi}{\partial x^j}\langle 1\rangle_{f_m} = 0$$

$$\rho_m' u^j_m + \rho_m u^{j\prime}_m + \frac{\partial}{\partial x^i}(\rho_m u^i_m u^j_m) + 4aH\rho_m u^j_m + \rho_m\frac{\partial\Psi}{\partial x^j} = 0$$

$$\cancel{\bar{\rho}_m'(1+\delta_m)u^j_m} + \cancel{\bar{\rho}_m}\delta_m' u^j_m + \cancel{\bar{\rho}_m}(1+\delta_m)u^{j\,\prime}_m + \frac{\partial}{\partial x^i}(\cancel{\bar{\rho}_m}(1+\delta_m)u^i_m u^j_m) +$$

$$+ \cancel{4} aH\cancel{\bar{\rho}_m}(1+\delta_m)u^j_m + \cancel{\bar{\rho}_m}(1+\delta_m)\frac{\partial\Psi}{\partial x^j} = 0$$

$$\cancel{\delta_m' u^j_m} + \cancel{(1+\delta_m)}u^{j\,\prime}_m - \cancel{\delta_m' u^j_m} + \cancel{(1+\delta_m)}u^i_m\frac{\partial u^j_m}{\partial x^i} + aH\cancel{(1+\delta_m)}u^j_m + \cancel{(1+\delta_m)}\frac{\partial\Psi}{\partial x^j} = 0$$

$$u^{j\,\prime}_m + u^i_m\frac{\partial u^j_m}{\partial x^i} + aHu^j_m + \frac{\partial\Psi}{\partial x^j} = 0, \quad (2)$$

$$\hookrightarrow \Delta\Psi = \frac{3}{2}\Omega_m a^2H^2\delta_m, \quad (3)$$

(1) Separate linear and nonlinear terms:

$$\underbrace{\delta_m' + \frac{\partial u^i_m}{\partial x^i}}_{\text{linear}} = \underbrace{-\delta_m\frac{\partial u^i_m}{\partial x^i} - u^i_m\frac{\partial\delta_m}{\partial x^i}}_{\text{nonlinear}\,\approx\,0} \approx 0$$

$\frac{\partial}{\partial x^j}$ (2)

$$\underbrace{\left(\frac{\partial u^j_m}{\partial x^j}\right)' + aH\frac{\partial u^j_m}{\partial x^j} + \frac{\partial^2\Psi}{\partial x^{j2}}}_{\text{linear}} = \underbrace{-u^i_m\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}u^j_m - \frac{\partial u^i_m}{\partial x^j}\frac{\partial u^j_m}{\partial x^i}}_{\text{nonlinear}} \approx 0$$

$$\Rightarrow \boxed{\delta_m'' + aH\delta_m' = \frac{3}{2}\Omega_m a^2H^2\delta_m} \qquad \left(\Omega_m = \frac{\bar{\rho}_m(\eta)}{\rho_c(\eta)}\right)$$

# Origin of perturbations

1.

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$$\nabla_x \delta p = \underbrace{\left(\frac{\partial \delta p}{\partial \delta \rho}\right)_{\delta S}}_{c_s^2} \nabla_x \delta \rho + \underbrace{\left(\frac{\partial \delta p}{\partial \delta S}\right)_{\delta \rho}}_{f(\bar{T})} \nabla_x \delta S$$

$$\Delta_x \delta p = c_s^2 \bar{\rho} \underbrace{\Delta_x \delta}_{-k^2\delta} + f(\bar{T}) \bar{S} \underbrace{\Delta_x \delta_s}_{-k^2 \delta_s} \qquad \delta = \frac{\delta \rho}{\bar{\rho}} \qquad \delta_s = \frac{\delta S}{\bar{S}}$$

$$\frac{\partial^2 \delta}{\partial t^2} + 2H \frac{\partial \delta}{\partial t} = \left(4\pi G \bar{\rho} - \frac{c_s^2 k^2}{a^2}\right)\delta - \frac{f(\bar{T}) \bar{S} k^2}{\bar{\rho} a^2} \delta_s$$

Initial conditions:

- isentropic perturbations $\rightarrow \delta_s = 0$ (pure density perturbations)
- isocurvature perturbations $\rightarrow \delta = 0$ (pure entropy perturbations)

$$\Downarrow$$

$$\Delta_x \delta \phi = 0$$

$\hookrightarrow$ No perturbations in the FLRW metric, perturbations in $\eta = \frac{\rho_b}{\rho_r}$

$\rightarrow$ Any perturbation can be written as a linear combination of both.

$\rightarrow$ If the evolution is adiabatic, isentropic perturbations remain isentropic.

$\rightarrow$ For isocurvature perturbations:

$$\delta \rho = \rho - \bar{\rho} = 0$$

$$\delta \rho = \rho_r + \rho_m - \bar{\rho}_r - \bar{\rho}_m = \bar{\rho}_r \delta_r + \bar{\rho}_m \delta_m = 0$$

$$\Rightarrow \frac{\delta_r}{\delta_m} = -\frac{\bar{\rho}_m}{\bar{\rho}_r} = -\frac{a}{a_0}$$

At early times: $a \simeq 0 \Rightarrow \bar{\rho}_m \ll \bar{\rho}_r \Rightarrow \delta_r \simeq 0 \Rightarrow \underbrace{\delta T \simeq 0}$

Hence the name „isothermal" perturbations. (only for $t \ll t_{eq}$!)

CMB is consistent with $\delta_s = 0$