The physics of recombination

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(a)

$$A + B \rightleftharpoons C$$

$F = k_B T \ln Z$ free energy

$$Z = \frac{Z_A^{N_A}}{N_A!} \cdot \frac{Z_B^{N_B}}{N_B!} \cdot \frac{Z_C^{N_C}}{N_C!}$$ canonical partition sum

Minimize F with respect to N_A!

$$\frac{\partial F}{\partial N_A} \stackrel{!}{=} \phi \Rightarrow \frac{\partial}{\partial N_A} \ln Z \stackrel{!}{=} \phi \Rightarrow \underbrace{\frac{\partial}{\partial N_A}\left(N_A \ln Z_A + N_B \ln Z_B + N_C \ln Z_C - \ln N_A! - \ln N_B! - \ln N_C!\right)}_{\phi}$$

Stirling's formula: $\ln N! = N \ln N - N + \mathcal{O}(\ln N)$

$$\frac{\partial}{\partial N}(\ln N!) \simeq \frac{\partial}{\partial N}(N \ln N - N) = \ln N + N\frac{1}{N} - 1 = \ln N$$

$$\frac{\partial N_B}{\partial N_A} = 1 \qquad \frac{\partial N_C}{\partial N_A} = -1 \qquad \left(\text{For: } \alpha A + \beta B \rightleftharpoons \gamma C \Rightarrow \frac{\partial N_B}{\partial N_A} = \frac{\beta}{\alpha} ; \frac{\partial N_C}{\partial N_A} = -\frac{\gamma}{\alpha}\right)$$

$$\Rightarrow \frac{\partial}{\partial N_A} \ln Z \simeq \ln Z_A + \ln Z_B - \ln Z_C - \ln N_A - \ln N_B + \ln N_C \stackrel{!}{=} \phi$$

$$\boxed{\frac{Z_A Z_B}{Z_C} = \frac{N_A N_B}{N_C}}$$ Law of mass action

(b)

$$Z_A = g_A\, e^{-\frac{m_A c^2}{k_B T}} \frac{V}{\lambda_A^3} \quad \rightarrow \quad \lambda_A = \frac{h}{\sqrt{2\pi m_A k_B T}}$$ thermal wavelength

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$$p^+ + e^- \rightleftharpoons H$$

Universe is neutral: $N_e = N_p$

Number of barions : $N_p + N_H = N_B$

$$N_H = N_B - N_e$$

$$x = \frac{N_e}{N_B}$$

$$\frac{Z_p Z_e}{Z_H} = \frac{N_p N_e}{N_H} = \frac{N_e^2}{N_B - N_e} = N_B \frac{x^2}{1-x}$$

$$\frac{g_p g_e}{g_H} V \left(\frac{\lambda_H}{\lambda_p \lambda_e}\right)^3 e^{-\frac{\overbrace{\overbrace{(m_p + m_e - m_H)}^{\Delta m} c^2}^{E_{binding}}}{k_B T}} = N_B \frac{x^2}{1-x}$$

$$\frac{g_p g_e}{g_H} \frac{1}{n_B} \left(\frac{2\pi k_B T}{h^2} \frac{m_p m_e}{m_H}\right)^{\frac{3}{2}} e^{-\frac{(m_p + m_e - m_H)c^2}{k_B T}} = \frac{x^2}{1-x}$$ Saha's equation

$n_B = \eta n_\gamma$ $\qquad$ $n_\gamma \rightarrow$ photon gas number density

$$n_\gamma = 16\pi \zeta(3) \left(\frac{k_B T}{hc}\right)^3$$

$\zeta(3) \rightarrow$ Riemann zeta function at $s=3$

$\zeta(3) \simeq 1.20206$

$\hookrightarrow g_p = 2$ $\qquad \frac{m_p}{m_H} \simeq 1$

$g_e = 2$

$g_H = 1$

$$4 \frac{1}{\eta} \frac{1}{16\pi \zeta(3)} \left(\frac{hc}{k_B T}\right)^3 \left(\frac{2\pi k_B T}{h^2}\right)^{\frac{3}{2}} m_e^{\frac{3}{2}} e^{-\frac{(m_p + m_e - m_H)c^2}{k_B T}} = \frac{x^2}{1-x}$$

$$\sqrt{\frac{\pi}{2}} \frac{1}{\eta} \frac{1}{\zeta(3)} m_e^{\frac{3}{2}} c^3 (k_B T)^{-\frac{3}{2}} e^{-\frac{(m_p + m_e - m_H)c^2}{k_B T}} = \frac{x^2}{1-x}$$

$$\underbrace{\sqrt{\frac{\pi}{2}} \frac{1}{\zeta(3)\eta} \cdot \left(\frac{m_e c^2}{k_B T}\right)^{\frac{3}{2}} e^{-\frac{(m_p + m_e - m_H)c^2}{k_B T}}}_{f(T)} = \frac{x^2}{1-x}$$

$$\frac{x^2}{1-x} = f$$

$$x^2 = f - fx$$

$$x^2 + fx - f = \emptyset$$

$$x = \frac{-f \pm \sqrt{f^2 + 4f}}{2} \rightarrow x = \frac{f\sqrt{1 + \frac{4}{f}} - f}{2}$$